Accelerating invasions along an environmental gradient

Gwenaël Peltier[1]

Contents

Abstract

We consider a population structured by a space variable and a phenotypical trait, submitted to dispersion, mutations, growth and nonlocal competition. This population is facing an environmental gradient: the optimal trait for survival depends linearly on the spatial variable. The survival or extinction depends on the sign of an underlying principal eigenvalue. We investigate the survival case when the initial data satisfies a so-called heavy tail condition in the space-trait plane. Under these assumptions, we show that the solution propagates in the favorable direction of survival by accelerating. We derive some precise estimates on the location of the level sets corresponding to the total population in the space variable, regardless of their traits. Our analysis also reveals that the orientation of the initial heavy tail is of crucial importance.

Key Words: structured population, nonlocal reaction-diffusion equation, propagation, accelerating fronts.

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[1]IMAG, Univ. Montpellier, CNRS, Montpellier, France. E-mail: gwenael.peltier@umontpellier.fr

1 Introduction

In this paper we study the propagation phenomena of the solution $n(t,x,y)$ to the following nonlocal parabolic Cauchy problem

$$\begin{cases} \partial_t n - \partial_{xx} n - \partial_{yy} n = \left(r(y-Bx) - \displaystyle\int_{\mathbb{R}} K(t,x,y,y') n(t,x,y') dy' \right) n, & t>0,\ (x,y)\in\mathbb{R}^2, \\ n(0,x,y) = n_0(x,y), & (x,y)\in\mathbb{R}^2. \end{cases} \tag{1}$$

We shall prove that if the initial data $n_0 \geq 0$ has a heavy tail, in a sense to be precised later, then any solution of (1) either goes extinct, or spreads in the favorable direction $y - Bx = 0$ by *accelerating*.

Equation (1) arises in some population dynamics models, see [21, 23]. In this context $n(t,x,y)$ represents a density of population at each time $t \geq 0$, structured by a space variable $x \in \mathbb{R}$ and a phenotypical trait $y \in \mathbb{R}$. This population is subject to four biological processes : migration, mutations, growth and competition. The diffusion operators $\partial_{xx} n$ and $\partial_{yy} n$ account for migration and mutations respectively. The growth rate of the population is given by $r(y-Bx)$, where r is negative outside a bounded interval. This corresponds to a population facing an environmental gradient : to survive at location x, an individual must have a trait close to the optimal trait $y_{opt} = Bx$ with $B > 0$. Thus, for invasion to occur, the population has to adapt during migration. As a consequence, it is expected that the population, if it survives, remains confined in a strip around the optimal line $y - Bx = 0$, where r is typically positive. Finally, we consider a logistic regulation of the population density that is local in the spatial variable and nonlocal in the trait. In other words, we consider that, at each location, there exists an intra-specific competition (for e.g. food) between all individuals, regardless of their traits.

The well-posedness of a Cauchy problem very similar to (1), but on a bounded domain, has been investigated in [23, Theorem I.1]. As mentioned in [1], we believe the arguments could be adapted in our context to show the existence of a global solution on an unbounded domain. A limiting argument would then provide the existence of solutions to (1) in the whole domain, thanks to the estimates on the tails of the solutions obtained in Lemma 3.4. However, in this article, our main interest lies in the qualitative properties of the solutions.

Survival vs extinction. As it is well known, the survival or extinction of the population (starting from a localized area) depends on the sign of the generalized principal eigenvalue λ_0 of the elliptic operator $-\partial_{xx} n - \partial_{yy} n - r(y-Bx)n$. If $\lambda_0 > 0$, the population goes extinct exponentially fast in time at rate $-\lambda_0$. If $\lambda_0 < 0$, the model, which is of Fisher-KPP type [17, 20], satisfies the Hair-Trigger Effect : any nonnegative initial data $n_0 \not\equiv 0$ leads to the survival of the population and its spreading to the whole space. Since we are concerned with propagation results, we shall assume that $\lambda_0 < 0$ and $n_0 \not\equiv 0$ in the rest of this introduction.

The local case. When the competition term in (1) is replaced by a local (in x and y) regulation, the equation satisfies the comparison principle and, moreover, one can assume $B = 0$ without loss of generality (through a rotation of coordinates). This yields to one of the models considered in [6], where the authors show the existence of travelling waves $\varphi(x - ct, y)$ solutions of the equation for speeds c greater than or equal to a critical value $c^* > 0$. Besides, there exists a unique positive stationary solution, which depends only on $y \in \mathbb{R}$, denoted here by $S(y)$, and all travelling waves connect the state $S(y)$ when $x \to -\infty$ to zero when $x \to +\infty$.

One of the main results in [6] concerns the Cauchy problem. When the initial data is compactly supported and satisfies $n_0(x,y) \leq S(y)$, the solution $n(t,x,y)$ converges locally uniformly in x towards $S(y)$, while propagating at speed c^* in both directions $x \to \pm\infty$.

The nonlocal but decoupled ($B = 0$) case. It is worth mentioning that the model (1) when $B = 0$ has been analyzed in [8]. In this context, one can decouple variables x and y, leading to a sequence of scalar Fisher-KPP equations, each obtained by projection on each eigenfunction of the elliptic operator $-\partial_{yy} - r(y)$. This technique [8] allows to prove, again, the existence of fronts $\varphi(x - ct, y)$ for speeds $c \geq c^* = 2\sqrt{-\lambda_0}$, as well as the survival of the population and its spreading at speed c^* for compactly supported initial data.

Akin to the local model in [6], there exists a unique positive stationary state $S = S(y)$ towards which the solution $n(t,x,y)$ converges. When $K \equiv 1$, the state S is actually the principal eigenfunction associated to λ_0

(with a unique choice of a multiplicative constant for S to be a positive steady state). For example, when r is quadratic, S is gaussian since it satisfies the equation of the harmonic oscillator.

The model (1). The propagation phenomenon of problem (1) has been investigated in [1, 3]. Notice that the authors in [1] also allow the environmental gradient to be shifted (say by Global Warming) at a given forced speed. Since $B \neq 0$, the decoupling argument [8] cannot be invoked here, which makes the analysis more involved. As far as travelling waves are concerned, problem (1) admits fronts of the form $\varphi(x-ct, y-Bx)$ only for speeds $c \geq c^* = 2\sqrt{\frac{-\lambda_0}{1+B^2}}$. However, the construction of those waves relies on a topological degree argument [3] and little is known about their behavior for $x \to -\infty$, with $y - Bx$ being constant. This is caused by the presence of a nonlocal competition term in (1), which prevents the equation to enjoy the comparison principle, see also [2, 9] for similar issues related to the scalar nonlocal Fisher-KPP equation.

The results [1] for the Cauchy problem (1) are the following : if n_0 is compactly supported, the total population at (t,x), given by $N(t,x) = \int_{\mathbb{R}} n(t,x,y)dy$, spreads at speed c^*. While the convergence of the solution towards a possible steady state remains an open question, it is worth pointing out that the population density remains mainly concentrated around the optimal trait $y = Bx$.

Accelerating invasions in the one-dimensional case. Before going further, let us here consider the scalar Fisher-KPP equation [17, 20], say, for simplicity,

$$\partial_t u - \partial_{xx} u = ru(1-u), \qquad t > 0,\, x \in \mathbb{R},$$

for some $r > 0$. A result from Hamel and Roques [19] shows that if the initial data displays a heavy tail, i.e. decays more slowly than any exponentially decaying function, then the population invades the whole space by accelerating. This is in contradistinction with the well-studied case of exponentially bounded initial data, where the level sets of the solution spread at finite speed, see [4, 24]. The authors also derive sharp estimates of the location, at large time, of these level sets [19].

Accelerating invasions in related models. Let us mention that acceleration also occurs for compactly supported initial data if the diffusion term is replaced with a fractional laplacian $-(-\partial_{xx})^\alpha u$, $0 < \alpha < 1$, see [15], or with a convolution term $J * u - u$ where the kernel $J = J(x)$ admits a heavy tail in both directions $x \to \pm\infty$ [18]. The latter case corresponds to models of population dynamics with long-distance dispersal.

The so-called cane-toad equation proposed in [5] is another biological invasion model where acceleration may occur. When the trait space is unbounded, propagation of the level sets of order $O(t^{3/2})$ has been predicted in [13]. This was then proved rigorously in [12] with a local competition term, and in [14] for both local and nonlocal (in trait) competition, using probabilistic and analytic arguments respectively. Notice however that acceleration is not induced here by initial heavy tails, but by a phenotype-dependent term before the spatial diffusion.

Accelerating invasions in model (1). Our aim is to study accelerating invasions in model (1), that is to determine if acceleration occurs if n_0 displays a "heavy tail", a notion that needs to be precised in the two-dimensional framework. First, we prove that if n_0 has a "heavy tail" in the favorable direction $y - Bx = 0$, then acceleration of the invasion occurs. Second, we derive precise estimates for the large-time location of the level sets of the solution. Finally, we also address the case where the "heavy tail" of n_0 is not positioned along the direction $y - Bx = 0$: in this case, since ill-directed, the heavy tail does not induce acceleration.

2 Assumptions and main results

2.1 Functions r, K and n_0

Throughout the paper, we make the following assumption.

Assumption 2.1. *The function $r(\cdot) \in L^\infty_{loc}(\mathbb{R})$ is* confining, *in the sense that, for all $\delta > 0$, there exists $R > 0$ such that*

$$r(z) \leq -\delta, \qquad \textit{for almost all } z \textit{ such that } |z| \geq R. \tag{2}$$

Additionally, there exists $r_{max} > 0$ such that $r(z) \leq r_{max}$ almost everywhere.

The function $K \in L^\infty((0,\infty) \times \mathbb{R}^3)$ satisfies

$$k_- \leq K \leq k_+, \qquad a.e.\ on\ (0,+\infty) \times \mathbb{R}^3, \tag{3}$$

for some $0 < k_- \leq k_+$.

Moreover, the initial data n_0 is non identically zero, and there exist $C_0 > 0$ and $\kappa_0 > 0$ such that

$$0 \leq n_0(x,y) \leq C_0 e^{-\kappa_0 |y - Bx|}, \qquad \text{for almost all } (x,y) \in \mathbb{R}^2. \tag{4}$$

An enlightening example of such function r is given by $r(z) = 1 - Az^2$, hence

$$r(y - Bx) = 1 - A(y - Bx)^2, \tag{5}$$

for some $A > 0$. Notice that the width of the strip where r is nonnegative, that is the favorable region, is $\frac{2}{\sqrt{A(1+B^2)}}$ (see Figure 1). As a result, both parameters A and $B > 0$ play a critical role to determine whether the population goes extinct or survives.

Condition (4) allows us to obtain estimates of the tails of $n(t,x,y)$ in the direction $y - Bx \to \pm\infty$ as given by Lemma 3.4. Note that condition (4) is not the aforementioned heavy tail condition, for any compactly supported function satisfies it. Before stating our heavy tail condition, we first need to consider some spectral problems.

2.2 Some eigenelements

As in [1, Section 4], rather than working in the (x,y) variables, let us write

$$n(t,x,y) = v(t,X,Y),$$

where X (resp. Y) represents the direction of (resp. the direction orthogonal to) the optimal trait $y = Bx$, that is

$$X = \frac{x + By}{\sqrt{1+B^2}}, \qquad Y = \frac{y - Bx}{\sqrt{1+B^2}}. \tag{6}$$

In these new variables, equation (1) is recast

$$\partial_t v - \partial_{XX} v - \partial_{YY} v = \left(\tilde{r}(Y) - \int_{\mathbb{R}} K(t,\chi,\psi,y') v\,(t,\chi,\psi)\,dy' \right) v, \tag{7}$$

where we use the shortcuts

$$\tilde{r}(Y) := r\left(\sqrt{1+B^2}Y\right), \qquad \chi = \chi(X,Y,y') := \frac{\frac{X-BY}{\sqrt{1+B^2}} + By'}{\sqrt{1+B^2}}, \qquad \psi = \psi(X,Y,y') := \frac{-B\frac{X-BY}{\sqrt{1+B^2}} + y'}{\sqrt{1+B^2}}.$$

We also note $v_0(X,Y) = n_0(x,y)$ the initial data in the new variables.

Next, as recalled in subsection 3.2, we are equipped with a generalized principal eigenvalue $\lambda_0 \in \mathbb{R}$ and a generalized principal eigenfunction $\Gamma_0 \in H^2_{loc}(\mathbb{R})$ satisfying

$$\begin{cases} -\partial_{YY}\Gamma_0(Y) - \tilde{r}(Y)\Gamma_0(Y) = \lambda_0 \Gamma_0(Y) & \text{for all } Y \in \mathbb{R}, \\ \Gamma_0 > 0, \qquad ||\Gamma_0||_{L^\infty(\mathbb{R})} = 1. \end{cases} \tag{8}$$

It is worth noting that, in the particular case where r is given by (5), expression (8) corresponds to the harmonic oscillator, for which these eigenelements can be explicitly computed as

$$\lambda_0 = \sqrt{A(1+B^2)} - 1, \qquad \Gamma_0(Y) = \exp\left(-\frac{1}{2}\sqrt{A(1+B^2)}Y^2\right). \tag{9}$$

Finally, for $R > 0$, let us consider λ_0^R, $\Gamma_0^R(Y)$ the principal eigenelements solving the Dirichlet problem on $(-R,R)$

$$\begin{cases} -\partial_{YY}\Gamma_0^R(Y) - \tilde{r}(Y)\Gamma_0^R(Y) = \lambda_0^R \Gamma_0^R(Y) & \text{for } Y \in (-R,R), \\ \Gamma_0^R(Y) = 0 & \text{for } Y = \pm R, \\ \Gamma_0^R(Y) > 0 & \text{for } Y \in (-R,R), \\ ||\Gamma_0^R||_\infty = 1. \end{cases} \tag{10}$$

As recalled in Proposition 3.3, there holds $\lambda_0^R \searrow \lambda_0$ as $R \to +\infty$.

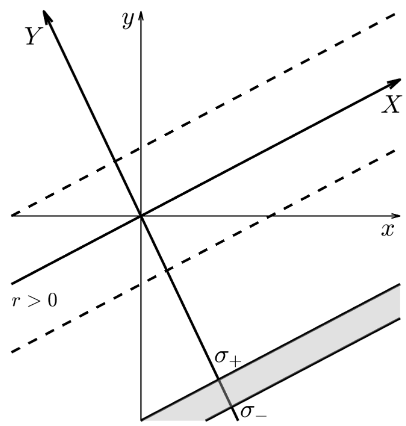

Figure 1: In grey, the region where $v_0(X,Y)$ is assumed to be greater than $\underline{u}_0(X)$, that is a one-dimensional heavy tail in the X direction. The dotted lines delimit the area where $r > 0$.

2.3 The heavy tail condition

We can now turn to the two-dimensional heavy tail condition. First, we recall or introduce some definitions for one-dimensional functions. A function $w\colon \mathbb{R} \to \mathbb{R}$ is said to be asymptotically front-like if

$$\liminf_{-\infty} w > 0, \quad w > 0, \quad \lim_{+\infty} w = 0. \tag{11}$$

A positive function $w\colon \mathbb{R} \to \mathbb{R}$ is said to have a heavy tail in $+\infty$ if

$$\lim_{x\to+\infty} w(x)e^{\varepsilon x} = +\infty, \qquad \forall \varepsilon > 0, \qquad \text{(1D Heavy Tail)}. \tag{12}$$

Typical examples are “lighter heavy tails” (13), algebraic tails (14), and “very heavy tails” (15), that is

$$w(x) \sim Ce^{-bx^a}, \quad \text{as } x \to +\infty, \quad \text{with } C, b > 0 \text{ and } a \in (0,1), \tag{13}$$
$$w(x) \sim Cx^{-a}, \quad \text{as } x \to +\infty, \quad \text{with } C, a > 0, \tag{14}$$
$$w(x) \sim C(\ln x)^{-a}, \quad \text{as } x \to +\infty, \quad \text{with } C, a > 0. \tag{15}$$

We now state our two-dimensional heavy tail condition for equation (1). Note that this condition is expressed in the new variables, thus it applies to v_0.

Assumption 2.2 (2D heavy tail condition)**.** *Let us consider the new coordinates (X,Y) given by (6). The initial data $v_0(X,Y) = n_0(x,y)$ is such that there exists $\underline{u}_0 \in L^\infty(\mathbb{R})$ satisfying (11)-(12), so that*

$$v_0(X,Y) \geq \underline{u}_0(X)\mathbf{1}_{[\sigma_-,\sigma_+]}(Y), \qquad \text{(2D Heavy Tail)}. \tag{16}$$

for some reals $\sigma_- < \sigma_+$.

Let us emphasize that we do not assume that $0 \in (\sigma_-,\sigma_+)$, meaning the initial data may not overlap the optimal trait line (see Figure 1). Moreover, the interval (σ_-,σ_+) may not only be arbitrarily far from zero, but also arbitrarily small. The key assumption is the correct orientation of the heavy tail, that is in the direction $X \to +\infty$, as highlighted by subsection 2.6. In the survival case $\lambda_0 < 0$ and under Assumption 2.2, we shall prove that the solution of (1) is accelerating.

We now aim at providing a precise estimate of the location of the level sets at large times. To do so, we first introduce the following definition.

Definition 2.3. A function w is said to satisfy the condition (Q) if

$$\begin{cases} w \in L^\infty(\mathbb{R}) \text{ and is uniformly continuous,} \\ \liminf_{-\infty} w > 0, \quad w > 0, \quad \lim_{+\infty} w = 0, \\ \exists \xi_0 \in \mathbb{R} \text{ such that } w \text{ is } C^2 \text{ and nonincreasing on } [\xi_0, +\infty), \\ w''(x) = o(w(x)) \text{ as } x \to +\infty, \end{cases} \tag{Q}$$

where o denotes the Landau symbol "little-o".

Notice that any function w satisfying (Q) also satisfies $w'(x) = o(w(x))$ as $x \to +\infty$, and thus displays a one-dimensional heavy tail in $+\infty$, see [19]. For the scalar Fisher-KPP equation, when the initial data satisfies (Q), the authors in [19] derived precise estimates on the location of the level sets of the solution. In our context, we make the following assumption on the initial data.

Assumption 2.4 ((Q)-Initial bounds)**.** *Let us consider the new coordinates (X, Y) given by (6). The initial data $v_0(X, Y) = n_0(x, y)$ is such that there exist functions $\overline{u}_0$, $\underline{u}_0$ satisfying (Q) so that*

$$\underline{u}_0(X)\mathbf{1}_{[\sigma_-,\sigma_+]}(Y) \le v_0(X, Y) \le \overline{u}_0(X)\Gamma_0(Y), \tag{17}$$

for some reals $\sigma_- < \sigma_+$.

In particular, if the initial data satisfies Assumption 2.4, then it satisfies Assumption 2.2. As far as the Y direction is concerned, when r is of the form (5), the eigenfunction Γ_0 is given by (9), so that (17) amounts to a gaussian control on the initial data. In the general case of a confining growth function (2), one can prove that $\Gamma_0(Y)$ decays at least exponentially when $|Y| \to +\infty$, see subsection 3.2. Under Assumption 2.4, we shall derive some precise estimates on the large-time position of the level sets, see Theorem 2.7.

2.4 The extinction case

As we shall see, under Assumption 2.1, the population either goes extinct or survives depending on the sign of the principal eigenvalue λ_0. In this short section we simply expose the result of [3], which covers the case $\lambda_0 > 0$.

Proposition 2.5 (Extinction case [3])**.** *Assume $\lambda_0 > 0$. Let r, K, n_0 satisfy Assumption 2.1. Suppose that there is $k > 0$ such that*

$$n_0(x, y) \le k\Gamma_0\left(\frac{y - Bx}{\sqrt{1 + B^2}}\right).$$

Then any global nonnegative solution of (1) *satisfies*

$$n(t, x, y) \le k\Gamma_0\left(\frac{y - Bx}{\sqrt{1 + B^2}}\right)e^{-\lambda_0 t}, \tag{18}$$

which implies $||n(t, \cdot, \cdot)||_{L^\infty(\mathbb{R}^2)} = O(e^{-\lambda_0 t})$, that is an exponentially fast extinction.

The proof of Proposition 2.5 is elementary as $n(t, x, y)$ and the right-hand side of (18) are respectively subsolution and supersolution of the parabolic operator $\partial_t n - \partial_{xx} n - \partial_{yy} n - r(y - Bx)n$. The maximum principle yields the result.

2.5 Main result : acceleration in the invasion case

We now investigate the case where the principal eigenvalue λ_0 is negative. In order to capture the spreading speed of the population in the space variable, we look at the evolution of the total population in (t, x), regardless of their trait. Thus, for any $\mu > 0$, we define the level set of n by

$$E^n_\mu(t) = \left\{ x \in \mathbb{R} \,\middle|\, \int_{\mathbb{R}} n(t, x, y)dy = \mu \right\}.$$

Let us emphasize again that, because of the nonlocal competition term, problem (1) does not enjoy the comparison principle. In such situation, and as mentioned in the introduction, the behavior "behind the front" is typically out of reach, see [1, 2, 3, 9, 16]. For such a reason, we are mainly interested in the spreading properties of $E_\mu^n(t)$ for *small* values μ.

Let us recall that under Assumption 2.1, if $\lambda_0 < 0$ and if $n_0 \not\equiv 0$ has compact support, then the population survives and the solution propagates at speed $c^* = 2\sqrt{\frac{-\lambda_0}{1+B^2}}$, see [1, Theorem 4.2]. Our first result shows that there is acceleration when, instead of being compactly supported, the initial data admits a heavy tail in the X direction, in the sense given by Assumption 2.2.

Theorem 2.6 (2D initial heavy tail implies acceleration)**.** *Assume $\lambda_0 < 0$. Let r, K, n_0 satisfy Assumptions 2.1 and 2.2. Let n be any global nonnegative solution of* (1)*. Then there exists $\beta > 0$ such that for any $\mu \in (0, \beta)$, there holds*

$$\frac{1}{t}\min E_\mu^n(t) \to +\infty, \qquad \text{as } t \to +\infty.$$

In other words, if the initial data is greater than or equal to a front-like function with a heavy tail in the direction $X \to +\infty$, the solution is accelerating. We will only give a sketch of the proof in subsection 4.4, as it is similar to the proof of Theorem 2.7 below.

We now state our main result, namely Theorem 2.7, which is an accurate estimate of the position of the accelerating level sets under Assumption 2.4. In the rest of this article, for any function $f\colon \mathbb{R} \to \mathbb{R}$, we denote $f^{-1}(a)$ the set $\{x \in \mathbb{R} \mid f(x) = a\}$.

Theorem 2.7 (Asymptotic position of the accelerating level sets)**.** *Assume $\lambda_0 < 0$. Let r, K, n_0 satisfy Assumptions 2.1 and 2.4. Let $R > 0$ be large enough such that $\lambda_0^R < 0$ (see subsection 2.2). Let n be any global nonnegative solution of* (1)*.*

Then there exists $\beta > 0$ so that for any $\mu \in (0, \beta)$, $\varepsilon \in (0, -\lambda_0^R)$, $\Gamma > 0$ and $\gamma > 0$, there exists $T^* = T^*_{\mu,\varepsilon,\gamma,\Gamma,R} \geq 0$ such that for all $t \geq T^*$, the set $E_\mu^n(t)$ is nonempty, compact, and satisfies

$$E_\mu^n(t) \subset \frac{1}{\sqrt{1+B^2}}\left[\min \underline{u}_0^{-1}\left(\Gamma e^{-(-\lambda_0^R-\varepsilon)t}\right), \max \overline{u}_0^{-1}\left(\gamma e^{-(-\lambda_0+\varepsilon)t}\right)\right]. \tag{19}$$

Let us make some comments on this theorem. Observe first that for t large enough there holds

$$\Gamma e^{-(-\lambda_0^R-\varepsilon)t} \in \left(0, \liminf_{-\infty} \underline{u}_0\right),$$
$$\gamma e^{-(-\lambda_0+\varepsilon)t} \in \left(0, \liminf_{-\infty} \overline{u}_0\right),$$

thus the sets $\underline{u}_0^{-1}\left(\Gamma e^{-(-\lambda_0^R-\varepsilon)t}\right)$ and $\overline{u}_0^{-1}\left(\gamma e^{-(-\lambda_0+\varepsilon)t}\right)$ are non-empty and bounded. Additionally, Assumption 2.4 implies that $\underline{u}_0 \leq C\overline{u}_0$ with $C = \min_{(\sigma_-,\sigma_+)} \Gamma_0 > 0$. In conjunction with $\lambda_0 < \lambda_0^R$, it follows that for t possibly even larger there holds

$$\min \underline{u}_0^{-1}\left(\Gamma e^{-(-\lambda_0^R-\varepsilon)t}\right) < \max \overline{u}_0^{-1}\left(\gamma e^{-(-\lambda_0+\varepsilon)t}\right),$$

giving a meaning to (19).

Next, notice that, given any two values μ and μ' in $(0, \beta)$, both level sets $E_\mu^n(t)$ and $E_{\mu'}^n(t)$ are included in the same interval given by expression (19). As a consequence, Theorem 2.7 implies that for any $\varepsilon \in (0, -\lambda_0^R)$ and positive real numbers γ and Γ, there holds

$$\liminf_{t\to+\infty} \inf_{x \leq (1+B^2)^{-1/2} \min \underline{u}_0^{-1}\left(\Gamma e^{-(-\lambda_0^R-\varepsilon)t}\right)} \int_{\mathbb{R}} n(t,x,y)dy \geq \beta,$$

$$\lim_{t\to+\infty} \sup_{x \geq (1+B^2)^{-1/2} \max \overline{u}_0^{-1}\left(\gamma e^{-(-\lambda_0+\varepsilon)t}\right)} \int_{\mathbb{R}} n(t,x,y)dy = 0.$$

The upper bound of $E_\mu^n(t)$ in (19) is valid for all levels μ, and only requires the upper bound of v_0 in Assumption 2.4. However, the lower bound of $E_\mu^n(t)$ is valid for levels $\mu < \beta$, and only requires the lower bound of v_0 in

Assumption 2.4. Also note that the lower bound in (19) leads to $\frac{1}{t}\min E^n_\mu(t) \to +\infty$ when $t \to +\infty$, thus we recover the acceleration.

We now give a sketch of the proof. The upper bound is much easier to prove since the nonlocal term is nonnegative. One constructs a supersolution of the form $\phi(t,X)\Gamma_0(Y)$ where ϕ satisfies $\partial_t\phi - \partial_{xx}\phi = (-\lambda_0)\phi$ with $\phi(0,\cdot)$ displaying a heavy tail. The upper bound of Lemma 3.1 is still valid in this case, which leads to the result with an adequate control of the tails of n.

The proof of the lower bound is much more involved. Suppose first that $[-R,R] \subset [\sigma_-,\sigma_+]$. Then, after bounding the nonlocal term with a refinement of a Harnack inequality, we construct a subsolution of the form $\underline{w}(t,X,Y) = u(t,X)\Gamma_0^R(Y)$ where u satisfies the Fisher-KPP equation. Therefore applying Lemma 3.1 allows us to conclude. Note that this might not be a subsolution if R were too small, leading to λ_0^R being possibly nonnegative. In the general case we may have $[-R,R] \not\subset [\sigma_-,\sigma_+]$. In that event we construct a subsolution $\underline{v}(t,X,Y)$ for $t \in [0,1]$, such that $\underline{v}(1,X,Y) \geq \rho \underline{u}_0(X)\Gamma_0^R(Y)$ on $\mathbb{R}\times[-R,R]$ for some $\rho > 0$. Then on $[1,+\infty)$ we consider a subsolution of the same form as $\underline{w}$, which gives the result.

In particular, to prove acceleration under the hypothesis $v_0(X,Y) \geq \underline{u}_0(X)\mathbf{1}_{[\sigma_-,\sigma_+]}(Y)$, we have to use Γ_0^R instead of Γ_0 in order to construct the subsolution. Because of this, we obtain $-\lambda_0^R$ in the lower bound of (19). Had we supposed the stronger hypothesis $v_0(X,Y) \geq \underline{u}_0(X)\Gamma_0(Y)$ instead, we could replace $-\lambda_0^R$ with $-\lambda_0$ and take any $\varepsilon \in (0,-\lambda_0)$. Let us also mention that β tends to zero as $R \to +\infty$, leading to a trade-off. Indeed, a large value of R provides a more precise location of the level sets, but also reduces the range of level sets being located.

We conclude this section by applying Theorem 2.7 in the cases where the functions $\underline{u}_0$ and $\overline{u}_0$ are of the forms (13)-(15). For simplicity, we only consider the lower bound.

Example 2.8. Suppose there exist $X_0, b > 0$ and $a \in (0,1)$ such that $\overline{u}_0(X) = Ce^{-bx^a}$ on $[X_0,+\infty)$. Then if we select $\Gamma = C$, the lower bound in (19) becomes

$$\min E^n_\mu(t) \geq \frac{1}{\sqrt{1+B^2}}\left(\frac{1}{b}(-\lambda_0^R - \varepsilon)t\right)^{1/a},$$

meaning the total population spreads with at least algebraic, superlinear speed.

Example 2.9. Suppose there exist $X_0, C, a > 0$ such that $\underline{u}_0(X) = CX^{-a}$ on $[X_0,+\infty)$. Then if we select $\Gamma = C$, the lower bound in (19) becomes

$$\min E^n_\mu(t) \geq \frac{1}{\sqrt{1+B^2}}\exp\left(\frac{1}{a}(-\lambda_0^R - \varepsilon)t\right),$$

thus the total population spreads with at least exponential speed.

Example 2.10. Suppose there exist $X_0 > 1$ and $C, a > 0$ such that $\overline{u}_0(X) = C(\ln x)^{-a}$ on $[X_0,+\infty)$. Then if we select $\Gamma = C$, the lower bound in (19) becomes

$$\min E^n_\mu(t) \geq \frac{1}{\sqrt{1+B^2}}\exp\left(\exp\left(\frac{1}{a}(-\lambda_0^R - \varepsilon)t\right)\right),$$

that is the total population spreads with at least superexponential speed.

2.6 When the heavy tail is ill-directed

When the initial data admits a heavy tail in direction $X \to +\infty$, in the sense of Assumption 2.2, Theorem 2.6 proves the acceleration of the propagation. It is worth wondering if acceleration still occurs when considering heavy tail initial condition in a different direction than $X \to +\infty$. For the sake of clarity, we only consider the direction $x \to +\infty$, but the proof is easily adapted to any direction

$$X' = \frac{x + B'y}{\sqrt{1+B'^2}} \to +\infty, \quad \text{with } B' \neq B.$$

Theorem 2.11 (Ill-directed heavy tail prevents acceleration)**.** *Suppose $\lambda_0 < 0$. Suppose r, K satisfy Assumption 2.1. Suppose n_0 satisfies*

$$0 \le n_0(x,y) \le u_0(x)\mathbf{1}_{[\sigma_-,\sigma_+]}(y), \tag{20}$$

where $u_0 \in L^\infty(\mathbb{R})$ and $\sigma_- < \sigma_+$. Let n be any global nonnegative solution of (1)*.*

Then if we define $c^* := 2\sqrt{\frac{-\lambda_0}{1+B^2}}$, there holds

$$\limsup_{t\to+\infty} \int_{\mathbb{R}} n(t,ct,y)dy = 0, \qquad \forall |c| > c^*. \tag{21}$$

Notice that u_0 appearing in (20) is only assumed to be bounded. In particular, even if $u_0 \equiv \mathrm{cst} > 0$, a much stronger assumption than a heavy tail, acceleration does not occur because of ill-orientation.

Before going further, let us mention that [1, Theorem 4.2] shows that, when r, K satisfy Assumption 2.1 and $n_0 \not\equiv 0$ is compactly supported, the spreading speed of the population is exactly c^*, in the sense that

$$\limsup_{t\to+\infty} \int_{\mathbb{R}} n(t,ct,y)dy = 0, \qquad \forall |c| > c^*, \tag{22}$$

$$\liminf_{t\to+\infty} \int_{\mathbb{R}} n(t,ct,y)dy \ge \beta, \qquad \forall |c| < c^*, \tag{23}$$

for some $\beta > 0$ that may depend on c when $|c| \to c^*$.

A consequence of Theorem 2.11 is that if $n_0 \not\equiv 0$ satisfies (20), the population spreads exactly at speed c^*, in the sense given by (22)-(23). To prove that (23) holds, one cannot invoke the comparison principle because of the nonlocal term in (1). However, an essential element of the proof of Theorem 2.11 is the control of the tails (50). Using it, one can adapt the proof of [1, Theorem 4.2] to show that (23) is valid.

Outline of the paper. The rest of this article is organized as follows. In Section 3 we provide some materials necessary to the proof, that is an equivalent of Theorem 2.7 for the scalar Fisher-KPP equation, some properties of functions satisfying (Q), some principal eigenelements of elliptic operators, some estimates on the tails of n as well as a refinement of the parabolic Harnack inequality. Section 4 is devoted to the proof of Theorem 2.7, and presents a sketch of the proof of Theorem 2.6. Finally, Section 5 addresses the proof of Theorem 2.11.

3 Preliminaries

3.1 Acceleration in the scalar Fisher-KPP equation

We consider here the Fisher-KPP equation with a logistic reaction term :

$$\begin{cases} \partial_t u - \partial_{xx} u = \Lambda u(1-u), & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{24}$$

with $\Lambda > 0$. The function $u_0 \colon \mathbb{R} \to [0,1]$ is assumed to be uniformly continuous and asymptotically front-like, in the sense of (11), and to display a (one-dimensional) heavy tail in $+\infty$, in the sense of (12). Under these assumptions, Hamel and Roques [19] proved that the level sets of u, defined for $\eta \in (0,1)$ by

$$E_\eta(t) := \{x \in \mathbb{R} \mid u(t,x) = \eta\},$$

propagate to the right by accelerating, that is $\min E_\eta(t)/t \to +\infty$ as $t \to +\infty$. Under assumption (Q), they also provide sharp estimates on the position of the level sets. This result, which will be an essential tool for our analysis, reads as follows.

Lemma 3.1 (See [19, Theorem 1.1])**.** *Let w satisfy (Q), see Definition 2.3. Let $u(t,x)$ be the solution of* (24) *with initial condition $u_0 := w/||w||_\infty$. Then for any $\eta \in (0,1)$, $\varepsilon \in (0,\Lambda)$, $\gamma > 0$ and $\Gamma > 0$, there exists $T = T_{\eta,\varepsilon,\gamma,\Gamma,\Lambda} \ge 0$ so that*

$$E_\eta(t) \subset w^{-1}\left(\left[\gamma e^{-(\Lambda+\varepsilon)t}, \Gamma e^{-(\Lambda-\varepsilon)t}\right]\right), \qquad \forall t \ge T.$$

In the sequel, in order to prove Theorem 2.7, we shall construct some sub- and super-solutions of the form $u(t,X)\Gamma_0(Y)$ where $u(t,X)$ solves (24) or a linear version of (24). Then, for the estimates on $E_\eta(t)$ provided by Lemma 3.1 to transfer to estimates on $E^n_\mu(t) = \left\{x \in \mathbb{R} \mid \int_{\mathbb{R}} n(t,x,y)dy = \mu\right\}$, we shall need a technical result which we now state.

Proposition 3.2. *Let w satisfy (Q). Then there exists $\xi_1 > \xi_0$ such that $w(x) > w(\xi_1)$ for any $x < \xi_1$.*

In addition, for any $0 < a < b$, $\Gamma_a > 0$, $\Gamma_b > 0$ and $\chi > 0$, there exists $t^* \geq 0$ such that

$$\min w^{-1}\left(\Gamma_a e^{-at}\right) + \chi \leq \min w^{-1}\left(\Gamma_b e^{-bt}\right), \qquad \forall t \geq t^*, \tag{25}$$

$$\max w^{-1}\left(\Gamma_a e^{-at}\right) + \chi \leq \max w^{-1}\left(\Gamma_b e^{-bt}\right), \qquad \forall t \geq t^*. \tag{26}$$

Proof. Set $m := \inf_{(-\infty,\xi_0]} w$. Since $\liminf_{-\infty} w > 0$ and $w > 0$, it is easy to check that $m > 0$. Now, since $\lim_{+\infty} w = 0$, there exists $x_+ > \xi_0$ such that $w(x_+) < m$. Since $w(x_+) > 0$ and $\lim_{+\infty} w = 0$, we can find $\xi_1 \geq x_+$ satisfying $w'(\xi_1) < 0$. Finally, as $w'(\xi_1) < 0$ and w is nonincreasing on $[\xi_0, +\infty)$, we can readily check that, for any $x < \xi_1$, there holds

$$w(\xi_1)\begin{cases} \leq w(x_+) < m \leq w(x), & \text{if } x \leq \xi_0, \\ < w(x), & \text{if } x \in (\xi_0, \xi_1), \end{cases}$$

which proves the first assertion.

We now turn to the second assertion. We only give a proof of (25), seeing as the proof of (26) is identical. In the first place, set $\bar{t} \geq 0$ large enough such that for any $t \geq \bar{t}$

$$\Gamma_a e^{-at}, \Gamma_b e^{-bt} \in (0, m), \qquad \forall t \geq \bar{t},$$

hence the sets $w^{-1}\left(\Gamma_a e^{-at}\right)$, $w^{-1}\left(\Gamma_b e^{-bt}\right)$ are well-defined and compact. Next, suppose by contradiction that there exist $0 < a < b$ and positive constants Γ_a, Γ_b, χ such that

$$\forall t^* \geq \bar{t}, \quad \exists t \geq t^* \qquad \min w^{-1}\left(\Gamma_a e^{-at}\right) + \chi > \min w^{-1}\left(\Gamma_b e^{-bt}\right).$$

As a result, we can construct an increasing sequence $(t_n)_n$ such that $\lim_{n\to+\infty} t_n = +\infty$ and the above inequality holds for $t = t_n$. In particular, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, there holds $\Gamma_a e^{-at_n} \leq w(\xi_1)$, whence

$$\min w^{-1}\left(\Gamma_a e^{-at_n}\right) \geq \xi_1 > \xi_0.$$

Meanwhile, since $b > a$, we can select N possibly even larger so that for any $n \geq N$ there holds

$$\Gamma_b e^{-bt_n} \leq \frac{\Gamma_a}{2} e^{-at_n} < \Gamma_a e^{-at_n}.$$

Both assertions imply, by monotony of w on $[\xi_0, +\infty)$, that

$$\min w^{-1}\left(\Gamma_a e^{-at_n}\right) < \min w^{-1}\left(\Gamma_b e^{-bt_n}\right).$$

Now, from the mean value theorem, there is $\theta_n \in \left(\min w^{-1}\left(\Gamma_a e^{-at_n}\right), \min w^{-1}\left(\Gamma_b e^{-bt_n}\right)\right)$ such that

$$w'(\theta_n) = \frac{\Gamma_a e^{-at_n} - \Gamma_b e^{-bt_n}}{\min w^{-1}\left(\Gamma_a e^{-at_n}\right) - \min w^{-1}\left(\Gamma_b e^{-bt_n}\right)} < 0,$$

therefore

$$|w'(\theta_n)| \geq \frac{\Gamma_a e^{-at_n} - \Gamma_b e^{-bt_n}}{\chi} \geq \frac{\Gamma_a e^{-at_n}}{2\chi}. \tag{27}$$

However, since w satisfies (Q), there holds $w'(x) = o(w(x))$ as $x \to +\infty$. As a consequence, there exists $x_\chi \in \mathbb{R}$ such that $|w'(x)| \leq \frac{1}{4\chi} w(x)$ for any $x \geq x_\chi$. As $\lim_{n\to+\infty} \theta_n = +\infty$, we obtain $\theta_n > x_\chi$ for n large enough. For such n, we derive the following inequality :

$$|w'(\theta_n)| \leq \frac{1}{4\chi} w(\theta_n) \leq \frac{1}{4\chi} \Gamma_a e^{-at_n},$$

which contradicts (27). Thus (25) holds. □

3.2 Some eigenvalue problems

We present here some useful eigenelements. This subsection is quoted from [1, Subsection 2.1], which was based on the results of [7, 10, 11].

The theory of generalized principal eigenvalue has been developed in [7], and is well adapted to our problem when r, thus $\tilde{r}$, is bounded. Following [7], we can then define, for $\tilde{r} \in L^\infty(\Omega)$ and $\Omega \subset \mathbb{R}$ a (possibly unbounded) nonempty domain, the generalized principal eigenvalue

$$\lambda(\tilde{r}, \Omega) := \sup\left\{\lambda \in \mathbb{R} \mid \exists \phi \in H^2_{loc}(\Omega),\, \phi > 0,\, \phi''(Y) + (\tilde{r}(Y) + \lambda)\phi(Y) \leq 0\right\}. \tag{28}$$

As shown in [7], if Ω is bounded, $\lambda(\tilde{r}, \Omega)$ coincides with the Dirichlet principal eigenvalue λ_D, that is the unique real number such that there exists ϕ defined on Ω (unique up to multiplication by a scalar) satisfying

$$\begin{cases} -\phi''(Y) - \tilde{r}(Y)\phi(Y) = \lambda_D \phi(Y) & \text{a.e. in } \Omega, \\ \phi > 0 & \text{on } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that $\lambda(\tilde{r}, \Omega) \leq \lambda(\tilde{r}, \Omega')$ if $\Omega \supset \Omega'$. The following proposition shows that $\lambda(\tilde{r}, \Omega)$ can be obtained as a limit of increasing domains.

Proposition 3.3. *Assume that $\tilde{r} \in L^\infty(\mathbb{R})$. For any nonempty domain $\Omega \subset \mathbb{R}$ and any sequence of nonempty domains $(\Omega_n)_{n\in\mathbb{N}}$ such that*

$$\Omega_n \subset \Omega_{n+1}, \qquad \cup_{n\in\mathbb{N}} \Omega_n = \Omega,$$

there holds $\lambda(\tilde{r}, \Omega_n) \searrow \lambda(\tilde{r}, \Omega)$ as $n \to +\infty$. Furthermore, there exists a generalized principal eigenfunction, that is a positive function $\Gamma \in H^2_{loc}(\mathbb{R})$ such that

$$-\Gamma''(Y) - \tilde{r}(Y)\Gamma(Y) = \lambda(\tilde{r}, \Omega)\Gamma(Y), \qquad \text{a.e. in } \Omega.$$

Since our growth function $\tilde{r}$ is only assumed to be bounded from above, we extend definition (28) to functions $\tilde{r}$ in $L^\infty_{loc}(\Omega)$ such that $\tilde{r} \leq r_{max}$ on Ω, for some $r_{max} > 0$. The set

$$\Lambda(\tilde{r}, \Omega) := \left\{\lambda \in \mathbb{R} \mid \exists \phi \in H^2_{loc}(\Omega),\, \phi > 0,\, \phi''(Y) + (\tilde{r}(Y) + \lambda)\phi(Y) \leq 0\right\}$$

is not empty since $\Lambda(\max(\tilde{r}, -r_{max}), \Omega) \subset \Lambda(\tilde{r}, \Omega)$, and is bounded from above, thanks to the monotony property of $\Omega \mapsto \Lambda(\tilde{r}, \Omega)$. Finally, going back to the proof of [7, Proposition 4.2], we notice that Proposition 3.3 remains valid under the weaker assumption $\tilde{r} \in L^\infty_{loc}(\Omega)$ is bounded from above.

It follows from the above discussion that we are equipped with the generalized principal eigenvalue $\lambda_0 \in \mathbb{R}$ and a generalized principal eigenfunction $\Gamma_0 \in H^2_{loc}(\mathbb{R})$ such that

$$\begin{cases} -\Gamma_0''(Y) - \tilde{r}(Y)\Gamma_0(Y) = \lambda_0 \Gamma_0(Y) & \text{a.e. in } \Omega, \\ \Gamma_0 > 0 & \text{on } \Omega, \\ ||\Gamma_0||_{L^\infty(\mathbb{R})} = 1. \end{cases}$$

Let us also mention that, given that $\tilde{r}$ satisfies Assumption 2.1, the function Γ_0 decays at least exponentially as $|Y| \to +\infty$. This result holds by using the comparison principle on $\{|Y| > Y_0\}$ with a supersolution of the form $Ce^{-a|Y|}$, with Y_0 large enough (so that $\tilde{r} + \lambda_0 \leq -\varepsilon$ for some $\varepsilon > 0$), a small enough and C large enough.

3.3 Preliminary estimates

The following lemma gathers preliminary results from [1], with n_0 satisfying Assumption 2.1 instead of being compactly supported. The proof of the following a priori estimates is easily adapted from [1, Lemmas 2.3 and 2.4] and is therefore omitted (see also the proof of Lemma 5.1).

Lemma 3.4 (Some a priori estimates)**.** *Let r, K, n_0 satisfy Assumption 2.1. Then, there exist $N_\infty > 0$, $C > 0$ and $\kappa > 0$ such that any global nonnegative solution of* (1) *satisfies*

$$\int_{\mathbb{R}} n(t, x, y)dy \leq N_\infty, \tag{29}$$

$$n(t,x,y) \leq Ce^{-\kappa|y-Bx|}, \tag{30}$$

for all $t \geq 0$, $x \in \mathbb{R}$, $y \in \mathbb{R}$.

While Lemma 3.4 provides us with a uniform bound, we need more precise estimates on the nonlocal term $\int_{\mathbb{R}} n(t,x,y)dy$. To do so, we invoke a refinement of the parabolic Harnack inequality, as exposed in [1].

For any $(t,x) \in (0,+\infty) \times \mathbb{R}^N$ with $N \geq 1$, we consider a solution $u(t,x)$ of the following linear parabolic equation

$$\partial_t u(t,x) - \sum_{i,j=1}^{N} a_{i,j}(t,x)\partial_{x_i x_j} u(t,x) - \sum_{i=1}^{N} b_i(t,x)\partial_{x_i} u(t,x) = f(t,x)u(t,x), \qquad t>0,\ x \in \mathbb{R}^N \tag{31}$$

where the coefficients are bounded, and $(a_{i,j})_{i,j=1,\dots,N}$ is uniformly elliptic.

Theorem 3.5 (A refinement of the Harnack inequality [1, Theorem 2.7])**.** *Assume that all the coefficients* $(a_{i,j})_{i,j=1,\dots,N}$*,* $(b_i)_{i=1,\dots,N}$*,* f *belong to* $L^{\infty}_{loc}((0,+\infty)\times\mathbb{R}^N)$*, and that* $(a_{i,j})$ *is uniformly positive definite on* $\mathbb{R}^N$*. Assume there exists* $K>0$ *such that, for all* $1 \leq i,j \leq N$*,*

$$a_{i,j}(t,x) \leq K, \;\; b_i(t,x) \leq K, \;\; f(t,x) \leq K, \qquad a.e. \text{ on } (0,+\infty)\times\mathbb{R}^N.$$

Let $R, \delta, U, \varepsilon, \rho$ *be positive constants.*

There exists $C>0$ *such that for any* $\bar{t} \geq \varepsilon$*, any* $\bar{x} \in \mathbb{R}^N$ *and any nonnegative weak solution* $u \in H^1((0,+\infty)\times\mathbb{R}^N)$ *of* (31) *satisfying* $||u||_{L^\infty(\mathbb{R}^N)} \leq U$*, there holds*

$$\max_{x\in B(\bar{x},R)} u(\bar{t},x) \leq C \min_{x\in B(\bar{x},R)} u(\bar{t},x) + \delta.$$

Notice that, as seen from the proof of [1, Theorem 2.7], the constant $C>0$ does not depend on $\bar{t}$ provided that $\bar{t} \geq \varepsilon > 0$, which validates the above setting.

4 Acceleration result

Subsections 4.1 and 4.2 are devoted to prove the following : under the hypotheses of Theorem 2.7, there exist $T^*_{\mu,\varepsilon,\gamma} > 0$ and $T^*_{\mu,\varepsilon,\Gamma,R} > 0$ such that

$$\int_{\mathbb{R}} n(t,x,y)dy < \mu, \qquad \forall x \geq (1+B^2)^{-1/2} \max \overline{u}_0^{-1}\left(\gamma e^{-(-\lambda_0+\varepsilon)t}\right), \qquad \forall t \geq T^*_{\mu,\varepsilon,\gamma}, \tag{32}$$

$$\int_{\mathbb{R}} n(t,x,y)dy > \mu, \qquad \forall x \leq (1+B^2)^{-1/2} \min \underline{u}_0^{-1}\left(\Gamma e^{-(-\lambda_0^R-\varepsilon)t}\right), \qquad \forall t \geq T^*_{\mu,\varepsilon,\Gamma,R}. \tag{33}$$

Subsection 4.3 concludes the proof of our main result, namely Theorem 2.7, based on (32)-(33). Lastly, in subsection 4.4, we sketch the proof of Theorem 2.6.

In the rest of this section, in view of (3), we shall consider $K \equiv 1$ without loss of generality. Additionally, to alleviate notations, the function $\tilde{r}(Y) = r(\sqrt{1+B^2}Y)$ will be denoted as r.

4.1 The upper bound (32)

This subsection is devoted to the proof of (32).

Lemma 4.1. *Let* $\Lambda > 0$ *and* ϕ *the solution of the Cauchy problem*

$$\begin{cases} \partial_t \phi - \partial_{XX}\phi = \Lambda\phi, & t>0,\ X \in \mathbb{R}, \\ \phi(0,X) = \overline{u}_0(X), & X \in \mathbb{R}, \end{cases} \tag{34}$$

where $\overline{u}_0$ *satisfies* (Q)*. Set* $E^{\phi}_{\eta}(t) = \{X \in \mathbb{R} \mid \phi(t,X) = \eta\}$ *for any* $\eta > 0$*.*

Then for any $\eta>0$*,* $\varepsilon \in (0,\Lambda)$*,* $\gamma > 0$*, there exists* $T = T_{\eta,\varepsilon,\gamma} \geq 0$ *such that for any* $t \geq T$ *the set* $E^{\phi}_{\eta}(t)$ *is nonempty, admits a maximum, and*

$$\max E^{\phi}_{\eta}(t) \leq \max \overline{u}_0^{-1}\left(\gamma e^{-(\Lambda+\varepsilon)t}\right).$$

This means that the upper bound of Theorem 3.1 is still valid when the logistic reaction term $\Lambda u(1-u)$ in (24) is replaced with Λu. Note that, as in Lemma 3.1, $T_{\eta,\varepsilon,\gamma}$ also depends on Λ. However, we ignore it here since we will fix $\Lambda = -\lambda_0$. The proof of Lemma 4.1 is easily adapted from that of [19, Theorem 1.1] and is consequently omitted.

Proof of (32). Set $\Lambda = -\lambda_0 > 0$, and

$$\overline{v}(t,X,Y) = \phi(t,X)\Gamma_0(Y),$$

where ϕ is the solution of the Cauchy problem (34). We readily check that $\overline{v}$ is a supersolution of the operator $\partial_t - \partial_{XX} - \partial_{YY} - r(Y)$:

$$\partial_t\overline{v} - \partial_{XX}\overline{v} - \partial_{YY}\overline{v} - r(Y)\overline{v} = \left(\partial_t\phi - \partial_{XX}\phi + \lambda_0\phi\right)\Gamma_0 = 0.$$

Meanwhile, since $v \geq 0$, it is clear that v is a subsolution of the same operator. Since (17) provides $v_0(X,Y) \leq \overline{v}(0,X,Y)$, we conclude with the maximum principle that $v \leq \overline{v}$ on $[0,+\infty)\times\mathbb{R}^2$.

Next, let $\mu > 0$, $\varepsilon \in (0,\Lambda)$, and $\gamma > 0$. One can select $\eta > 0$ and $\delta > 0$ such that

$$\eta\sqrt{1+B^2}\int_{\mathbb{R}}\Gamma_0(Y)dY + \delta < \mu. \tag{35}$$

From our control of the tails (30), there exist $C,\kappa > 0$ such that

$$v(t,X,Y) \leq Ce^{-\kappa\sqrt{1+B^2}|Y|}, \qquad \forall t\geq 0,\ \forall (X,Y)\in\mathbb{R}^2.$$

From there, we can find $\zeta = \zeta(\delta) > 0$ large enough, such that for any $t \geq 0$ and $x\in\mathbb{R}$, there holds

$$\begin{aligned}
\int_{\mathbb{R}} n(t,x,y)dy &= \int_{\mathbb{R}} v\Big(t, X(x,y), Y(x,y)\Big)dy \\
&= \int_{\mathbb{R}} v\left(t, \frac{x+By}{\sqrt{1+B^2}}, \frac{y-Bx}{\sqrt{1+B^2}}\right)dy \\
&= \sqrt{1+B^2}\int_{\mathbb{R}} v(t,\sqrt{1+B^2}x+Bs,s)ds \\
&\leq \sqrt{1+B^2}\left[\int_{-\infty}^{-\zeta} Ce^{-\kappa\sqrt{1+B^2}|s|}ds + \int_{-\zeta}^{+\infty}\overline{v}(t,\sqrt{1+B^2}x+Bs,s)ds\right] \\
&\leq \delta + \sqrt{1+B^2}\int_{-\zeta}^{+\infty}\phi(t,\sqrt{1+B^2}x+Bs)\Gamma_0(s)ds.
\end{aligned}$$

Now, set $E_\eta^\phi(t) = \{x\in\mathbb{R} \mid \phi(t,X)=\eta\}$. We will show that for any

$$x \geq \frac{1}{\sqrt{1+B^2}}\max\overline{u}_0^{-1}\left(\gamma e^{-(\Lambda+\varepsilon)t}\right),$$

and for t large enough there holds $\phi(t,\sqrt{1+B^2}x+Bs) \leq \eta$ for any $s\geq -\zeta$. For now, let us only assume the condition on x. By applying Proposition 3.2 to $\overline{u}_0$ with

$$\begin{cases} a = \Lambda+\varepsilon/2, & b = \Lambda+\varepsilon, \\ \Gamma_a = \gamma, & \Gamma_b = \gamma, \\ \chi = B\zeta, \end{cases}$$

there exists $t^*(a,b,\Gamma_a,\Gamma_b,\chi) = t^*_{\varepsilon,\gamma,\zeta} \geq 0$ such that for all $t\geq t^*_{\varepsilon,\gamma,\zeta}$ there holds

$$\max\overline{u}_0^{-1}\left(\gamma e^{-(\Lambda+\varepsilon/2)t}\right) + B\zeta \leq \max\overline{u}_0^{-1}\left(\gamma e^{-(\Lambda+\varepsilon)t}\right).$$

From there, Lemma 4.1 proves the existence of $T_{\eta,\varepsilon/2,\gamma} \geq 0$ such that for all $t \geq T^*_{\mu,\varepsilon,\gamma} := \max(t^*_{\varepsilon,\gamma,\zeta}, T_{\eta,\varepsilon/2,\gamma})$ the following holds

$$\begin{aligned}\max E^\phi_\eta(t) &\leq \max \overline{u}_0^{-1}\left(\gamma e^{-(\Lambda+\varepsilon/2)t}\right)\\ &\leq \max \overline{u}_0^{-1}\left(\gamma e^{-(\Lambda+\varepsilon)t}\right) - B\zeta\\ &\leq \sqrt{1+B^2}x - B\zeta.\end{aligned}$$

It is then easily deduced that for any $s > -\zeta$, there holds $\phi(t, \sqrt{1+B^2}x + Bs) < \eta$. Indeed, assume by contradiction that there exists $x_0 > \sqrt{1+B^2}x - B\zeta$ such that $\phi(t, x_0) \geq \eta$. Since for any $t \geq 0$ one has $\phi(t, X) \to 0$ as $X \to +\infty$, there would exist $x_1 \in E^\phi_\eta(t) \cap [x_0, +\infty)$, which contradicts the above inequality.

Finally, for any $t \geq T^*_{\mu,\varepsilon,\gamma}$ and $x \geq \frac{1}{\sqrt{1+B^2}} \max \overline{u}_0^{-1}\left(\gamma e^{-(\Lambda+\varepsilon)t}\right)$, there holds :

$$\begin{aligned}\int_{\mathbb{R}} n(t,x,y)dy &\leq \delta + \eta\sqrt{1+B^2}\int_{-\zeta}^{+\infty} \Gamma_0(s)ds\\ &\leq \delta + \eta\sqrt{1+B^2}\int_{\mathbb{R}} \Gamma_0(s)ds,\end{aligned}$$

which, combined with (35), proves (32). □

4.2 The lower bound (33)

This subsection is devoted to the proof of (33).

Lemma 4.2. *Let $\underline{u}_0$ satisfy (Q), see Definition 2.3. Then there exists a function $\underline{\underline{u}}_0 : \mathbb{R} \to \mathbb{R}$ such that*

- $\underline{\underline{u}}_0 \leq \underline{u}_0$,
- *there exists $\xi_2 \in \mathbb{R}$ such that $\underline{\underline{u}}_0 = \underline{u}_0$ on $[\xi_2, +\infty)$,*
- *$\underline{\underline{u}}_0$ satisfies (Q),*
- *$\underline{\underline{u}}_0$ is of class C^2 on $\mathbb{R}$ and there exists $K \geq 0$ such that $|\underline{\underline{u}}_0''| \leq K\underline{\underline{u}}_0$ on $\mathbb{R}$.*

Proof of Lemma 4.2. From Proposition 3.2, there exists $\xi_1 > \xi_0$ so that $\underline{u}_0(X) \geq \underline{u}_0(\xi_1)$ for any $X \leq \xi_1$. Fix $h \in (0, \xi_1 - \xi_0)$.

Since $\underline{u}_0$ satisfies (Q), there is $\xi_2 > \xi_1$ such that $\underline{u}_0(\xi_2) < \underline{u}_0(\xi_1)$ and $|\underline{u}_0''(X)| \leq \underline{u}_0(X)$ for all $X \geq \xi_2$. Next, one can construct a nondecreasing, concave function $\phi \colon \mathbb{R}_+ \to \mathbb{R}_+$ of class C^2 satisfying

$$\phi(x) = \begin{cases} x, & \forall x \leq \underline{u}_0(\xi_2),\\ m, & \forall x \geq \underline{u}_0(\xi_1),\end{cases}$$

for some $m < \underline{u}_0(\xi_1)$. Finally, set $\underline{\underline{u}}_0 = \phi \circ \underline{u}_0$. Let us prove that $\underline{\underline{u}}_0$ satisfies all the desired properties.

Given that ϕ is concave, we have $\phi(x) \leq x$ on $\mathbb{R}_+$, thus $\underline{\underline{u}}_0 \leq \underline{u}_0$ on $\mathbb{R}$. Meanwhile, for any $X \geq \xi_2$, there holds $\underline{u}_0(X) \leq \underline{u}_0(\xi_2)$, which implies $\underline{\underline{u}}_0(X) = \underline{u}_0(X)$. Furthermore, the function $\underline{\underline{u}}_0$ is of class C^2 on $\mathbb{R}$. Indeed, by composition $\underline{\underline{u}}_0$ is C^2 on $(\xi_1 - h, +\infty)$, whereas on $(-\infty, \xi_1]$, $\underline{\underline{u}}_0$ is constant from our choice of ξ_1.

Let us check that the function $\underline{\underline{u}}_0$, which is clearly bounded, satisfies condition (Q), see Definition 2.3. Given that ϕ and $\underline{u}_0$ are uniformly continuous, so is $\underline{\underline{u}}_0$. Then, since $\underline{\underline{u}}_0 = \underline{u}_0$ on $[\xi_2, +\infty)$ and $\underline{u}_0$ satisfies (Q), we collect for free the properties corresponding to the third and fourth lines of condition (Q), as well as $\lim_{+\infty} \underline{\underline{u}}_0 = 0$. Meanwhile, $\phi > 0$ leads to $\underline{\underline{u}}_0 > 0$. Eventually, given our choice of ξ_1, one deduces $\liminf_{-\infty} \underline{u}_0 \geq \underline{u}_0(\xi_1)$, whence $\liminf_{-\infty} \underline{\underline{u}}_0 = m > 0$. As a result, $\underline{\underline{u}}_0$ does satisfy (Q).

It remains to prove the existence of a real $K \geq 0$ such that $|\underline{\underline{u}}_0''| \leq K\underline{\underline{u}}_0$ on $\mathbb{R}$. From our choice of ξ_2, there holds

$$|\underline{\underline{u}}_0''(X)| \begin{cases} = 0 < \underline{\underline{u}}_0(X) & \forall X \leq \xi_1,\\ = |\underline{u}_0''(X)| \leq \underline{u}_0(X) = \underline{\underline{u}}_0(X) & \forall X \geq \xi_2.\end{cases}$$

Meanwhile, $\underline{\underline{u}}_0$ is C^2 and positive on $[\xi_1, \xi_2]$, thus there exists $K' \geq 0$ such that for any X in $[\xi_1, \xi_2]$

$$|\underline{\underline{u}}_0''(X)| \leq K' \min_{[\xi_1,\xi_2]} \underline{\underline{u}}_0 \leq K' \underline{\underline{u}}_0(X).$$

Thus it suffices to choose $K = \max(1, K')$. □

We now turn to the derivation of estimate (33).

Proof of (33). The proof involves three steps. First we construct a subsolution $\underline{v}$ on $[0,1] \times \mathbb{R} \times [\sigma_- - \alpha, \sigma_+ + \alpha]$, which, for $\alpha > 0$ large enough, provides a lower bound of the form $v(1, X, Y) \geq \rho \underline{\underline{u}}_0(X) \Gamma_0^R(Y)$ for some $\rho > 0$, where $\underline{\underline{u}}_0$ is constructed from $\underline{u}_0$ as in Lemma 4.2.

Next, equipped with this lower bound provided by $\underline{v}$, we construct a second subsolution $\underline{w}$ on $[1, +\infty) \times \mathbb{R}^2$, which spreads and accelerates in the direction $X \to +\infty$. Let us recall that due to the nonlocal term, equation (1) does not satisfy the comparison principle. Thus, to prove that $v(t, \cdot, \cdot) \geq \underline{w}(t, \cdot, \cdot)$ for $t \geq 1$, we invoke a refinement of the parabolic Harnack inequality, that is Theorem 3.5.

The last step consists in transferring the estimates on the level sets of $\underline{w}$ into estimates on $E_\mu^n(t)$, similarly to the proof of (32) in subsection 4.1.

First subsolution. Let $\alpha > 0$ large enough so that $[-R, R] \subset (\sigma_- - \alpha, \sigma_+ + \alpha)$. Let $p(t, Y)$ be the solution of the initial boundary value problem

$$\begin{cases} \partial_t p - \partial_{YY} p = 0, & t \in (0,1],\ Y \in (\sigma_- - \alpha, \sigma_+ + \alpha), \\ p(t, Y) = 0, & t \in (0,1],\ Y = \sigma_\pm \pm \alpha, \\ p(0, Y) = p_0(Y), & Y \in (\sigma_-, \sigma_+), \end{cases}$$

with p_0 the quadratic polynomial that satisfies $p_0(\sigma_\pm) = 0$ and $p_0\left(\frac{\sigma_+ + \sigma_-}{2}\right) = 1$. In other words, $p(t, Y)$ solves the one-dimensional heat equation on $[0,1] \times [\sigma_- - \alpha, \sigma_+ + \alpha]$, with zero Dirichlet conditions imposed on the boundary. Let us recall that N_∞, C and κ are positive real numbers such that (29) and (30) hold. Moreover, since $r \in L^\infty_{loc}(\mathbb{R})$, there exists $r_{min} \leq 0$ such that $r(Y) \geq r_{min}$ for any $Y \in [\sigma_- - \alpha, \sigma_+ + \alpha]$. We define the following subsolution

$$\underline{v}(t, X, Y) := e^{-kt} \underline{\underline{u}}_0(X) p(t, Y), \tag{36}$$

$$k := K - r_{min} + N_\infty, \tag{37}$$

$$\Omega := \{(t, X, Y) \mid 0 < t < 1,\ X \in \mathbb{R},\ Y \in (\sigma_- - \alpha, \sigma_+ + \alpha)\}, \tag{38}$$

where $\underline{\underline{u}}_0$ is constructed from $\underline{u}_0$ as in Lemma 4.2, with the associated constant $K \geq 0$, so that $k \geq N_\infty > 0$.

Let us prove that $v \geq \underline{v}$ on $\overline{\Omega}$. We first check that $v \leq \underline{v}$ on the parabolic boundary of Ω, that is

$$\partial_p \Omega = \Big(\{0\} \times \mathbb{R} \times [\sigma_-, \sigma_+]\Big) \bigcup D_+ \bigcup D_-,$$

where

$$D_\pm := \big\{(t, X, \sigma_\pm \pm \alpha) \mid t \in (0,1], X \in \mathbb{R}\big\}.$$

On the one hand, it follows from (17) that $v_0(X, Y) \geq \underline{v}(0, X, Y)$ for $X \in \mathbb{R}$ and $Y \in [\sigma_-, \sigma_+]$. On the other hand, on $D_+ \cup D_-$, one has $\underline{v} = 0 \leq v$. Thus $v \geq \underline{v}$ on $\partial_p \Omega$. It remains to show that $v - \underline{v}$ is a supersolution of a parabolic problem on Ω. First, for $t \in (0,1)$, there holds

$$\begin{aligned} \partial_t v - \partial_{XX} v - \partial_{YY} v &= r(Y) v - v \int_{\mathbb{R}} v(t, \chi, \psi) dy \\ &\geq (r_{min} - N_\infty) v. \end{aligned}$$

Meanwhile, since $|\underline{\underline{u}}_0''| \leq K \underline{\underline{u}}_0$, one obtains

$$\begin{aligned} \partial_t \underline{v} - \partial_{XX} \underline{v} - \partial_{YY} \underline{v} &= e^{-kt} \left[-k \underline{\underline{u}}_0 p - \underline{\underline{u}}_0'' p\right] \\ &= e^{-kt} \left[(r_{min} - N_\infty) \underline{\underline{u}}_0 p + (-K \underline{\underline{u}}_0 - \underline{\underline{u}}_0'') p\right] \\ &\leq (r_{min} - N_\infty) \underline{v}. \end{aligned}$$

We conclude by the maximum principle that $v \geq \underline{v}$ on $\overline{\Omega}$. We have thus completed the first step of the proof.

Second subsolution. We now turn to the construction of a second subsolution. The maximum principle shows that $p(1,Y)$ is positive on $(\sigma_- - \alpha, \sigma_+ + \alpha)$. Since $[-R,R] \subset (\sigma_- - \alpha, \sigma_+ + \alpha)$, there exists $p_{min} > 0$ so that $p(1,Y) \geq p_{min}$ for any $Y \in [-R,R]$. Thus, for any $X \in \mathbb{R}$ and $Y \in [-R,R]$, there holds

$$\begin{aligned} v(1,X,Y) &\geq \underline{v}(1,X,Y) \\ &\geq e^{-k} p_{min} \underline{\underline{u}}_0(X) \\ &\geq e^{-k} p_{min} \underline{\underline{u}}_0(X) \Gamma_0^R(Y), \end{aligned}$$

where Γ_0^R solves (10). Fix now any real number $\rho > 0$ small enough so that $\rho < ||\underline{\underline{u}}_0||_\infty e^{-k} p_{min}$. We allow ourselves to take ρ even smaller if needed. Set $\Lambda_\varepsilon^R = -\lambda_0^R - \varepsilon/2 > 0$. For any $t \geq 1$ and $X, Y \in \mathbb{R}$, define

$$\underline{w}(t,X,Y) \coloneqq \rho u(t,X) \Gamma_0^R(Y),$$

where $u(t,X)$ solves the Fisher-KPP equation

$$\begin{cases} u_t - u_{XX} = \Lambda_\varepsilon^R u(1-u), & t > 1, \ X \in \mathbb{R}, \\ u(1,X) = \underline{\underline{u}}_0(X)/||\underline{\underline{u}}_0||_\infty, & X \in \mathbb{R}. \end{cases}$$

In particular, $u(t,X) \in [0,1]$ thanks to the maximum principle.

We shall now prove that $v(t,X,Y) > \underline{w}(t,X,Y)$ on $[1,+\infty) \times \mathbb{R}^2$. Given our choice of ρ, we indeed have $v(1,X,Y) > \underline{w}(1,X,Y)$ on $\mathbb{R}^2$. Assume by contradiction that the closed set

$$E = \{t > 1 \mid \exists (X,Y) \in \mathbb{R}^2, \quad v(t,X,Y) = \underline{w}(t,X,Y)\},$$

is nonempty. Set $t_0 \coloneqq \min E > 1$ and $(X_0,Y_0) \in \mathbb{R}^2$ the point where $v(t_0,X_0,Y_0) = \underline{w}(t_0,X_0,Y_0)$. Note that this implies $Y_0 \in (-R,R)$, since the maximum principle yields $v(t,\cdot,\cdot) > 0$ for any $t > 0$. Before going further, we first use Theorem 3.5 to estimate the nonlocal term in (7). Let (x_0,y_0) be the corresponding coordinates of (X_0,Y_0) obtained through the change of variable (6). Fix $M > 0$, large enough so that $\frac{3C}{\kappa} e^{-\kappa M} \leq \varepsilon/8$ and $|y_0 - Bx_0| = \sqrt{1+B^2}|Y_0| \leq M$. Thanks to the control of the tails (30), there holds

$$\begin{aligned} \int_{\mathbb{R}} v(t_0, \chi(X_0,Y_0,y), \psi(X_0,Y_0,y)) dy &= \int_{\mathbb{R}} v\left(t_0, \frac{\frac{X_0 - BY_0}{\sqrt{1+B^2}} + By}{\sqrt{1+B^2}}, \frac{-B\frac{X_0-BY_0}{\sqrt{1+B^2}} + y}{\sqrt{1+B^2}}\right) dy \\ &= \int_{\mathbb{R}} n(t_0,x_0,y) dy = \int_{\mathbb{R}} n(t_0,x_0,Bx_0+y) dy \\ &\leq 2M \max_{y \in [-M,M]} n(t_0,x_0,Bx_0+y) + \int_{[-M,M]^c} Ce^{-\kappa|y|} dy. \end{aligned} \tag{39}$$

Next, in order to estimate the first term of (39), let us recall that the solutions are uniformly bounded, as implied by (30). This allows us to use the refinement of the Harnack inequality, namely Theorem 3.5, with $\delta = \frac{C}{2M\kappa} e^{-\kappa M} > 0$. Thus there exists a constant $C_M > 0$ such that

$$\begin{aligned} \max_{(x,y) \in [-M,M]^2} n(t_0, x_0 + x, Bx_0 + y) &\leq C_M \min_{(x,y) \in [-M,M]^2} n(t_0, x_0 + x, Bx_0 + y) + \delta \\ &\leq C_M n(t_0,x_0,y_0) + \delta, \end{aligned}$$

which we plug into (39) to obtain

$$\int_{\mathbb{R}} v\left(t_0, \frac{\frac{X_0-BY_0}{\sqrt{1+B^2}} + By'}{\sqrt{1+B^2}}, \frac{-B\frac{X_0-BY_0}{\sqrt{1+B^2}} + y'}{\sqrt{1+B^2}}\right) dy' \leq 2MC_M v(t_0,X_0,Y_0) + \frac{3C}{\kappa} e^{-\kappa M}.$$

Going back to our proof by contradiction, since $(\underline{w} - v)$ is negative on $[1,t_0) \times \mathbb{R}^2$, it reaches its maximum on $[1,t_0] \times \mathbb{R}^2$ at the point (t_0,X_0,Y_0). Thus

$$\left[\partial_t(\underline{w} - v) - \partial_{XX}(\underline{w} - v) - \partial_{YY}(\underline{w} - v) - r(Y_0)(\underline{w} - v)\right](t_0,X_0,Y_0) \geq 0, \tag{40}$$

On the one hand, there holds

$$\begin{aligned}
-\left[\partial_t v - \partial_{XX} v - \partial_{YY} v - r(Y)v\right](t_0, X_0, Y_0) &= v(t_0, X_0, Y_0)\int_{\mathbb{R}} v(t_0, \chi, \psi) dy \\
&\leq 2MC_M \underline{w}(t_0, X_0, Y_0)^2 + \frac{3C}{\kappa} e^{-\kappa M} \underline{w}(t_0, X_0, Y_0).
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
\partial_t \underline{w} - \partial_{XX} \underline{w} - \partial_{YY} \underline{w} - r(Y)\underline{w} &= \rho\left(\partial_t u - \partial_{XX} u + \lambda_0^R u\right)\Gamma_0^R \\
&\leq \rho\left(\Lambda_\varepsilon^R u + \lambda_0^R u\right)\Gamma_0^R \\
&\leq -\frac{\varepsilon}{2}\underline{w}.
\end{aligned}$$

Thus, (40) leads to

$$\begin{aligned}
0 &\leq -\frac{\varepsilon}{2}\underline{w}(t_0, X_0, Y_0) + 2MC_M \underline{w}(t_0, X_0, Y_0)^2 + \frac{3C}{\kappa} e^{-\kappa M} \underline{w}(t_0, X_0, Y_0) \\
&\leq \left[-\frac{\varepsilon}{2} + 2MC_M \rho + \frac{3C}{\kappa} e^{-\kappa M}\right] \underline{w}(t_0, X_0, Y_0) \\
&\leq \left[-\frac{\varepsilon}{2} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8}\right] \underline{w}(t_0, X_0, Y_0),
\end{aligned}$$

provided we select ρ small enough so that $2MC_M\rho \leq \varepsilon/8$. Note that reducing ρ may change the values of t_0, X_0, Y_0 but all above estimates remain true since there always holds $Y_0 \in (-R, R)$ and C_M does not depend on $t_0 \geq 1$. In the end, one has $-\varepsilon\underline{w}(t_0, X_0, Y_0)/4 \geq 0$. This implies $\underline{w}(t_0, X_0, Y_0) \leq 0$, which is absurd. As a result, $\underline{w}(t, X, Y) < v(t, X, Y)$ for any $t \geq 1$ and $X, Y \in \mathbb{R}$.

Conclusion. Before going further, let us mention that, for any $t > 1$ and $X \in \mathbb{R}$, there holds $u(t, X) < 1$, thus

$$\int_{\mathbb{R}} w(t, X, Y) dy < \rho \int_{\mathbb{R}} \Gamma_0^R(Y) dy = \rho\sqrt{1 + B^2} \int_{\mathbb{R}} \Gamma_0^R(y) dy =: \beta.$$

Therefore, the lower bound $v(t, X, Y) \geq \underline{w}(t, X, Y)$ on $[1, +\infty) \times \mathbb{R}^2$ does not provide any information on the location of $E_\mu^n(t)$ for levels $\mu \geq \beta$. As a result the location of larger levels are seemingly out of reach, which is typical of equations without comparison principle, as already mentioned in the introduction and subsection 2.5.

Now, given any $\mu \in (0, \beta)$, one can select $\eta = \eta(\mu) \in (0, 1)$ such that $\eta > \mu/\beta$. Since $v \geq \underline{w}$ on $[1, +\infty) \times \mathbb{R}^2$, there holds for any $t \geq 1$

$$\begin{aligned}
\int_{\mathbb{R}} n(t, x, y) dy &\geq \int_{\mathbb{R}} \underline{w}(t, X, Y) dy \\
&\geq \int_{\mathbb{R}} \rho u(t, X) \Gamma_0^R(Y) dy \\
&\geq \rho \int_{\mathbb{R}} u\left(t, \frac{x + By}{\sqrt{1 + B^2}}\right) \Gamma_0^R\left(\frac{y - Bx}{\sqrt{1 + B^2}}\right) dy \\
&\geq \rho\sqrt{1 + B^2} \int_{\mathbb{R}} u(t, \sqrt{1 + B^2} x + Bs) \Gamma_0^R(s) ds \\
&\geq \rho\sqrt{1 + B^2} \int_{-R}^{R} u(t, \sqrt{1 + B^2} x + Bs) \Gamma_0^R(s) ds.
\end{aligned}$$

We will show that for any

$$x \leq \frac{1}{\sqrt{1 + B^2}} \min \underline{u}_0^{-1}\left(\Gamma e^{-(\Lambda_\varepsilon^R - \varepsilon/2)t}\right),$$

and for t large enough there holds $u(t, \sqrt{1+B^2}x + Bs) \geq \eta$ for $|s| \leq R$. For now, let us only assume the condition on x. By applying Proposition 3.2 to $\underline{u}_0$ with

$$\begin{cases} a = \Lambda_\varepsilon^R - \varepsilon/2, & b = \Lambda_\varepsilon^R - \varepsilon/4, \\ \Gamma_a = \Gamma, & \Gamma_b = \Gamma, \\ \chi = BR, \end{cases}$$

we deduce that there exists $t^*_{a,b,\Gamma_a,\Gamma_b,\chi} = t^*_{\Gamma,R,\varepsilon} \geq 0$ such that for any $s \in [-R, R]$ and $t \geq t^*_{\Gamma,R,\varepsilon}$, there holds

$$\begin{aligned} \sqrt{1+B^2}x + Bs &\leq \min \underline{u}_0^{-1}\left(\Gamma e^{-(\Lambda_\varepsilon^R - \varepsilon/2)t}\right) + Bs \\ &\leq \min \underline{u}_0^{-1}\left(\Gamma e^{-(\Lambda_\varepsilon^R - \varepsilon/2)t}\right) + BR \\ &\leq \min \underline{u}_0^{-1}\left(\Gamma e^{-(\Lambda_\varepsilon^R - \varepsilon/4)t}\right). \end{aligned}$$

Since $\eta \in (0, 1)$, Lemma 3.1 gives the existence of a real $T_{\eta,\varepsilon/4,\Gamma,\Lambda_\varepsilon^R} \geq 0$ such that

$$\min E_\eta(t) \geq \min \underline{\underline{u}}_0^{-1}\left(\Gamma e^{-(\Lambda_\varepsilon^R - \varepsilon/4)t}\right), \qquad \forall t \geq T_{\eta,\varepsilon/4,\Gamma,\Lambda_\varepsilon^R},$$

where $E_\eta(t) = \{x \in \mathbb{R} \mid u(t,x) = \eta\}$. However, Lemma 4.2 provides some $\xi_2 \in \mathbb{R}$ such that $\underline{\underline{u}}_0(X) = \underline{u}_0(X)$ for $X \geq \xi_2$, and since $\underline{\underline{u}}_0$ satisfies (Q) (see Definition 2.3), one can take $T_{\eta,\varepsilon/4,\Gamma,\Lambda_\varepsilon^R}$ possibly even larger so that $\min \underline{\underline{u}}_0^{-1}\left(\Gamma e^{-(\Lambda_\varepsilon^R - \varepsilon/4)t}\right) \geq \xi_2$. Also, since $\underline{\underline{u}}_0 \leq \underline{u}_0$, there holds $\min \underline{u}_0^{-1}\left(\Gamma e^{-(\Lambda_\varepsilon^R - \varepsilon/4)t}\right) \geq \xi_2$, so that

$$\min \underline{\underline{u}}_0^{-1}\left(\Gamma e^{-(\Lambda_\varepsilon^R - \varepsilon/4)t}\right) = \min \underline{u}_0^{-1}\left(\Gamma e^{-(\Lambda_\varepsilon^R - \varepsilon/4)t}\right), \qquad \forall t \geq T_{\eta,\varepsilon/4,\Gamma,\Lambda_\varepsilon^R}.$$

Additionally, seeing that $\inf_{X \leq 0} u(t, X) \to 1$ as $t \to +\infty$, there is $\boldsymbol{t}_{\eta,R,\varepsilon} \geq 0$ such that

$$\liminf_{X \to -\infty} u(t, X) > \eta, \qquad \forall t \geq \boldsymbol{t}_{\eta,R,\varepsilon}.$$

Finally, set

$$T^*_{\eta,\varepsilon,\Gamma,R} \coloneqq \max(t^*_{\Gamma,R,\varepsilon},\, T_{\eta,\varepsilon/4,\Gamma,\Lambda_\varepsilon^R},\, \boldsymbol{t}_{\eta,R,\varepsilon}),$$

then for any $x \leq \frac{1}{\sqrt{1+B^2}} \min \underline{u}_0^{-1}\left(\Gamma e^{-(\Lambda_\varepsilon^R - \varepsilon/2)t}\right)$ and $t \geq T^*_{\eta,\varepsilon,\Gamma,R}$, there holds

$$\begin{aligned} \min E_\eta(t) &\geq \min \underline{u}_0^{-1}\left(\Gamma e^{-(\Lambda_\varepsilon^R - \varepsilon/4)t}\right) \\ &\geq \sqrt{1+B^2}x + BR. \end{aligned} \tag{41}$$

Thus for any $s \in (-R, R)$, one has $u(t, \sqrt{1+B^2}x + Bs) > \eta$. Indeed, assume by contradiction that there exists $s_0 \in (-R, R)$ such that $u(t, \sqrt{1+B^2}x + Bs_0) \leq \eta$. Given that $t \geq \boldsymbol{t}_{\eta,R,\varepsilon}$, there would exist $x_1 \in E_\eta(t) \cap (-\infty, \sqrt{1+B^2}x + Bs_0)$, which is absurd considering (41).

Consequently, whenever $t \geq T^*_{\eta,\varepsilon,\Gamma,R}$ and $x \leq \frac{1}{\sqrt{1+B^2}} \min \underline{u}_0^{-1}\left(\Gamma e^{-(\Lambda_\varepsilon^R - \varepsilon/2)t}\right)$, there holds

$$\begin{aligned} \int_{\mathbb{R}} n(t,x,y)dy &\geq \rho\sqrt{1+B^2}\int_{-R}^{R} \eta \Gamma_0^R(s)ds \\ &\geq \eta\beta > \mu. \end{aligned}$$

Since $\Lambda_\varepsilon^R - \varepsilon/2 = -\lambda_0^R - \varepsilon$, this concludes the proof of (33). □

4.3 Conclusion

Proof of Theorem 2.7. Let us recall that (32)-(33) have been established in subsections 4.1 and 4.2 respectively. We can now complete the proof of Theorem 2.7. Set

$$N_t(x) := \int_{\mathbb{R}} n(t,x,y)dy,$$

for any $t \geq 0$ and $x \in \mathbb{R}$. Set $T^*_{\mu,\varepsilon,\gamma,\Gamma,R} := \max(T^*_{\mu,\varepsilon,\gamma}, T^*_{\mu,\varepsilon,\Gamma,R})$. One can reformulate (32)-(33) as follows : there exists $\beta > 0$ so that for any $\mu \in (0,\beta)$, $\varepsilon \in (0, -\lambda_0^R)$, $\Gamma > 0$ and $\gamma > 0$, there exists $T^*_{\mu,\varepsilon,\gamma,\Gamma,R} \geq 0$ such that for all $t \geq T^*_{\mu,\varepsilon,\gamma,\Gamma,R}$, there holds

$$\begin{aligned} N_t(x) < \mu, \qquad & \forall x \geq \frac{1}{\sqrt{1+B^2}} \max \overline{u}_0^{-1}\left(\gamma e^{-(-\lambda_0+\varepsilon)t}\right), \\ N_t(x) > \mu, \qquad & \forall x \leq \frac{1}{\sqrt{1+B^2}} \min \underline{u}_0^{-1}\left(\Gamma e^{-(-\lambda_0^R-\varepsilon)t}\right), \end{aligned}$$

Since N_t is continuous for any $t > 0$, and $E_\mu^n(t) = N_t^{-1}(\mu)$, the level set $E_\mu^n(t)$ is nonempty, closed, and included in

$$\frac{1}{\sqrt{1+B^2}}\left[\min \underline{u}_0^{-1}\left(\Gamma e^{-(-\lambda_0^R-\varepsilon)t}\right), \max \overline{u}_0^{-1}\left(\gamma e^{-(-\lambda_0+\varepsilon)t}\right)\right],$$

for any $t \geq T^*_{\mu,\varepsilon,\gamma,\Gamma,R}$. This concludes the proof of Theorem 2.7. □

4.4 A heavy tail induces acceleration

In this short subsection, we give a sketch of the proof of Theorem 2.6, as it follows the same lines as the proof of (33) done in subsection 4.2.

By assumption, there holds $\lambda_0 < 0$. Select $R > 0$ large enough so that $\lambda_0^R < 0$ (see subsection 2.2). Choose any $c > 2\sqrt{-\lambda_0^R}$ and set $\alpha_c \in (0, \sqrt{-\lambda_0^R})$ the only real satisfying $c = \alpha_c + \frac{-\lambda_0^R}{\alpha_c}$. Since there exists $m > 0$ such that $\underline{u}_0(X) \geq \min(m, e^{-\alpha_c X})$, we can construct, as in the proof of Lemma 4.2, a C^2 function $\underline{\underline{u}}_0 : \mathbb{R} \to [0,1]$ that satisfies

- $\underline{\underline{u}}_0 \leq \underline{u}_0$,
- $\underline{\underline{u}}_0$ is asymptotically front-like, i.e. satisfies (11),
- $\underline{\underline{u}}_0(X) = e^{-\alpha_c X}$ for X large enough,
- there exists $K \geq 0$ such that $|\underline{\underline{u}}_0''| \leq K\underline{\underline{u}}_0$ on $\mathbb{R}$.

Then, select $\alpha > 0$ large enough so that $[-R,R] \subset (\sigma_- - \alpha, \sigma_+ + \alpha)$. Set $\underline{v}$ as in (36)-(38). From (16), the maximum principle allows us to conclude that $v \geq \underline{v}$ on $[0,1] \times \mathbb{R}^2$.

Next, as in subsection 4.2, we can construct a second subsolution on $[1,+\infty) \times \mathbb{R}^2$ of the form $\underline{w}(t,X,Y) = \rho u(t,X)\Gamma_0^R(Y)$ for some $\rho > 0$, where Γ_0^R solves (10) and $u(t,X)$ solves

$$\begin{cases} u_t - u_{XX} = -\lambda_0^R u(1-u), & t > 1,\ X \in \mathbb{R}, \\ u(1,X) = \underline{\underline{u}}_0(X), & X \in \mathbb{R}. \end{cases}$$

However, since $\underline{\underline{u}}_0(X)$ decays as $e^{-\alpha_c X}$, the function $u(t,X)$ converges to a shift of the front $\varphi_c(X-ct)$ solution of the Fisher-KPP equation $u_t - u_{XX} = -\lambda_0^R u(1-u)$, see [24]. As a consequence, with the same calculation as in subsection 4.2, we deduce the existence of a level $\beta > 0$ such that for any $\mu \in (0,\beta)$, there holds

$$\liminf_{t\to+\infty}\left(\frac{1}{t}\min E_\mu^n(t)\right) \geq \frac{1}{\sqrt{1+B^2}}c,$$

Since $c > 2\sqrt{-\lambda_0^R}$ may be chosen arbitrarily large, there holds $\min E_\mu^n(t)/t \to +\infty$ as $t \to +\infty$, for all $\mu \in (0,\beta)$, leading to the result of Theorem 2.6.

5 No acceleration for ill-directed heavy tails

In contradistinction with Section 4, we will keep the notation $\tilde{r}(Y) = r(y - Bx) = r(\sqrt{1+B^2}Y)$, since the two coordinate systems (x, y) and (X, Y) will be used in conjunction during the proof.

Proof of Theorem 2.11. We only give the proof for $c > c^*$, as the case $c < -c^*$ is similar. We first set the following positive constants

$$s^* := c^* \sqrt{1+B^2} = 2\sqrt{-\lambda_0}, \qquad \gamma := \sqrt{-\lambda_0}.$$

Fix any $s > s^*$, and define

$$\psi(t,x,y) := Ce^{-\gamma(X-st)}\Gamma_0(Y),$$
$$\varphi(t,x,y) := e^{-\alpha t}u(t,x)p(t,y),$$

where X, Y are given by (6), $\alpha \geq \gamma^2\left(1+\frac{1}{B^2}\right)$ may be chosen arbitrarily large, $C > 0$ is a positive constant to be determined later, and the functions $u(t,x)$, $p(t,y)$ respectively solve

$$\begin{cases} \partial_t u - \partial_{xx} u = u(||u_0||_\infty - u) & t > 0,\, x \in \mathbb{R}, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{42}$$

$$\begin{cases} \partial_t p - \partial_{yy} p = 0 & t > 0,\, y \in \mathbb{R}, \\ p(0,y) = \mathbf{1}_{[\sigma_-,\sigma_+]}(y) & y \in \mathbb{R}. \end{cases} \tag{43}$$

Since p solves the one-dimensional heat equation, it is expressed as the convolution

$$p(t,y) = \frac{1}{\sqrt{4\pi t}} \int_{\sigma_-}^{\sigma_+} e^{-(y-z)^2/(4t)} dz. \tag{44}$$

We shall prove that $\psi + \varphi \geq n$ using the maximum principle. Notice that one clearly has $\psi(0,\cdot,\cdot) \geq 0$, and $\varphi(0,x,y) \geq n_0(x,y)$ with (20). Also, since nonnegative, n is a subsolution of the linear local operator :

$$\mathcal{L} := \partial_t - \partial_{xx} - \partial_{yy} - r(y - Bx),$$
$$= \partial_t - \partial_{XX} - \partial_{YY} - \tilde{r}(Y),$$

thus it suffices to prove that $\mathcal{L}(\psi+\varphi) \geq 0$ on $(0,+\infty) \times \mathbb{R}^2$ to conclude.

Given that $-\partial_{YY}\Gamma_0 - \tilde{r}(Y)\Gamma_0(Y) = \lambda_0\Gamma_0(Y)$, we have

$$\mathcal{L}\psi = \left(\gamma s - \gamma^2 + \lambda_0\right)\psi > 0,$$

since $\gamma s^* - \gamma^2 + \lambda_0 = 0$. In particular, ψ is a supersolution.

Let us now turn our attention to the function φ. Since r satisfies Assumption 2.1, there exists $Y_0 > 0$ such that $\tilde{r}(Y) \leq -\alpha$ whenever $|Y| \geq Y_0$. Set

$$\Omega_0 := \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; |Y| = \frac{1}{\sqrt{1+B^2}}|y - Bx| < Y_0 \right\}.$$

On Ω_0^c, there holds $-\tilde{r}(Y) = -r(y - Bx) \geq \alpha$. Therefore this implies

$$\mathcal{L}\varphi = -\alpha\varphi + (||u_0||_\infty - u)\varphi - r(y-Bx)\varphi \geq (||u_0||_\infty - u)\varphi \geq 0.$$

Thus $\mathcal{L}(\psi+\varphi) \geq 0$ on $(0,+\infty) \times \Omega_0^c$.

Next, let us consider the domain Ω_0. On this domain, φ may no longer be a supersolution because r may be greater than $-\alpha$. However we shall prove that $\psi + \varphi$ is a supersolution. Indeed, there holds

$$\mathcal{L}\psi \geq Q_1 e^{-\gamma(X-st)}, \tag{45a}$$

$$\mathcal{L}\varphi \geq -Q_2 e^{-\alpha t}p(t,y), \tag{45b}$$

where

$$Q_1 := \left(\gamma s - \gamma^2 + \lambda_0\right) C \min_{|Y|\leq Y_0} \Gamma_0(Y) > 0, \tag{46a}$$

$$Q_2 := (\alpha + r_{max})||u_0||_\infty. \tag{46b}$$

Now, let us divide Ω_0 into two parts :

$$\Omega_- := \Omega_0 \cap \{(x,y) \mid BX - Y_0 \leq \theta\},$$
$$\Omega_+ := \Omega_0 \cap \{(x,y) \mid BX - Y_0 > \theta\},$$

where

$$\theta := \max\left(1, (\sigma_+ + 1)\sqrt{1+B^2}\right) > 0.$$

On the domain Ω_-, one has

$$\mathcal{L}\psi \geq Q_1 e^{-\gamma X} \geq Q_1 e^{-\gamma(Y_0+\theta)/B}$$
$$\mathcal{L}\varphi \geq -Q_2 e^{-\alpha t}||p(0,\cdot)||_\infty \geq -Q_2.$$

In view of (46) it suffices to take C large enough to reach $\mathcal{L}(\psi+\varphi) \geq 0$ on $(0,+\infty)\times\Omega_-$.

It remains to prove that $\mathcal{L}(\psi+\varphi) \geq 0$ on $(0,+\infty)\times\Omega_+$. For any $(x,y)\in\Omega_+$, we have

$$y = \frac{BX+Y}{\sqrt{1+B^2}} \geq \frac{BX-Y_0}{\sqrt{1+B^2}} \geq \frac{\theta}{\sqrt{1+B^2}} > \sigma_+.$$

Consequently, from (44) we obtain

$$\begin{aligned} p(t,y) &\leq \frac{1}{\sqrt{4\pi t}}(\sigma_+ - \sigma_-)e^{-(y-\sigma_+)^2/(4t)} \\ &= \frac{1}{\sqrt{4\pi t}}(\sigma_+ - \sigma_-)\exp\left(-\frac{1}{4t}\left(\frac{BX+Y}{\sqrt{1+B^2}} - \sigma_+\right)^2\right) \\ &\leq \frac{1}{\sqrt{4\pi t}}(\sigma_+ - \sigma_-)e^{-Z^2/4t}, \end{aligned}$$

where

$$Z := \frac{BX - Y_0}{\sqrt{1+B^2}} - \sigma_+ \geq \frac{\theta}{\sqrt{1+B^2}} - \sigma_+ \geq 1.$$

As a result, from (45), there holds

$$\begin{aligned} \mathcal{L}(\psi+\varphi) &\geq Q_1 e^{-\gamma X} e^{\gamma s t} - Q_2 e^{-\alpha t} p(t,y) \\ &\geq \widetilde{Q_1} e^{-\gamma Z\sqrt{1+1/B^2}} e^{\gamma s t} - \widetilde{Q_2} t^{-1/2} e^{-\alpha t} e^{-Z^2/4t}, \end{aligned}$$

where

$$\widetilde{Q_1} := Q_1 e^{-\gamma\left(Y_0 + \sqrt{1+B^2}\sigma_+\right)/B},$$
$$\widetilde{Q_2} := Q_2(\sigma_+ - \sigma_-)\frac{1}{\sqrt{4\pi}}.$$

Thus $\mathcal{L}(\psi+\varphi)\geq 0$ on $(0,+\infty)\times\Omega_+$ if

$$\widetilde{Q_1} e^{-\gamma Z\sqrt{1+B^2}/B} \geq \widetilde{Q_2} t^{-1/2} e^{-Z^2/4t} e^{(-\alpha-\gamma s)t}, \tag{47}$$

for every $Z \geq 1$ and $t > 0$. For any such Z, the function

$$g\colon t \mapsto t^{-1/2} e^{-Z^2/4t} e^{(-\alpha-\gamma s)t},$$

defined on $(0,+\infty)$, attains its maximum at

$$t_{max} = \frac{\sqrt{1+4(\gamma s+\alpha)Z^2}-1}{4(\gamma s+\alpha)} > 0,$$

which, since $\sqrt{1+a^2}-1 \le a \le \sqrt{1+a^2}$ for any $a \ge 0$, leads to

$$\begin{aligned} g(t) &\le Q_3 \exp\left(-\frac{(\gamma s+\alpha)Z^2}{\sqrt{1+4(\gamma s+\alpha)Z^2}-1}\right)\exp\left(-\frac{\sqrt{1+4(\gamma s+\alpha)Z^2}}{4}\right) \\ &\le Q_3 e^{-\sqrt{\gamma s+\alpha}Z}, \end{aligned}$$

with $Q_3 := t_{max}^{-1/2} e^{1/4} > 0$. Therefore, (47) amounts to

$$\widetilde{Q_1} e^{-\gamma Z\sqrt{1+1/B^2}} \ge \widetilde{Q_2} Q_3 e^{-\sqrt{\gamma s+\alpha}Z}.$$

Now, let us recall that since $\alpha \ge \gamma^2\left(1+\frac{1}{B^2}\right)$, we have

$$\gamma\sqrt{1+\frac{1}{B^2}} \le \sqrt{\gamma s+\alpha}.$$

Finally, by increasing $C>0$, if necessary, one has $\widetilde{Q_1} \ge \widetilde{Q_2} Q_3$, thus $\mathcal{L}(\psi+\varphi) \ge 0$ on $(0,+\infty)\times\Omega_+$.

Putting all together, we have thus proved $\mathcal{L}(\psi+\varphi)\ge 0$ on $(0,+\infty)\times\mathbb{R}^2$ and, from the comparison principle, $n(t,x,y) \le \psi(t,x,y)+\varphi(t,x,y)$ on $(0,+\infty)\times\mathbb{R}^2$. In other words, for any $s>s^*$ and $\alpha \ge \gamma^2\left(1+\frac{1}{B^2}\right)$, there exists $C>0$ such that for any $t>0$ and $(x,y)\in\mathbb{R}^2$, there holds

$$n(t,x,y) \le Ce^{-\gamma(X-st)}\Gamma_0(Y) + e^{-\alpha t}u(t,x)p(t,y). \tag{48}$$

We are now in the position to complete the proof of (21). This requires an additional control of the tails of n which is postponed to Lemma 5.1. In the sequel, we select $C'>0$ and $\kappa>0$ such that (50) holds. Choose any $c>c^* = s^*/\sqrt{1+B^2}$ and $s\in(s^*, c\sqrt{1+B^2})$. Now, fix $\mu>0$ and select $\zeta>0$ large enough so that

$$C'\int_{-\infty}^{-\zeta} e^{-\kappa|y|}dy < \frac{\mu}{2}.$$

Additionally, there exists $T_{\alpha,\mu}\ge 0$ such that for any $t\ge T_{\alpha,\mu}$ and for all $x\in\mathbb{R}$, there holds

$$\int_{\mathbb{R}} e^{-\alpha t}u(t,x)p(t,y)dy \le e^{-\alpha t}||u_0||_\infty(\sigma_+-\sigma_-) \le \frac{\mu}{2}.$$

Then, set $\xi := -\zeta + Bct$. For any $t\ge T_{\alpha,\mu}$, combining (48) and (50) one obtains

$$\begin{aligned} \int_{\mathbb{R}} n(t,ct,y)dy &\le \int_{-\infty}^{\xi}\left[C'e^{-\kappa|y-Bct|}+e^{-\alpha t}u(t,ct)p(t,y)\right]dy + \int_{\xi}^{+\infty}\left[\varphi(t,ct,y)+\psi(t,ct,y)\right]dy \\ &\le \int_{-\infty}^{-\zeta} C'e^{-\kappa|y|}dy + \int_{\mathbb{R}} e^{-\alpha t}u(t,ct)p(t,y)dy + \int_{\xi}^{+\infty}\psi(t,ct,y)dy \\ &\le \mu + C\int_{\xi}^{+\infty}\exp\left(-\gamma\left(\frac{ct+By}{\sqrt{1+B^2}}-st\right)\right)\Gamma_0\left(\frac{y-Bct}{\sqrt{1+B^2}}\right)dy \\ &\le \mu + C\sqrt{1+B^2}\int_{-\zeta/\sqrt{1+B^2}}^{+\infty} e^{-\gamma\left(\sqrt{1+B^2}ct+Bz-st\right)}\Gamma_0(z)dz \\ &\le \mu + \widetilde{C}e^{-\gamma\left(\sqrt{1+B^2}c-s\right)t} \end{aligned}$$

where $\widetilde{C} = C\sqrt{1+B^2}e^{\gamma B\zeta/\sqrt{1+B^2}}\int_{\mathbb{R}}\Gamma_0(z)dz > 0$. From there we deduce $\limsup_{t\to+\infty}\int_{\mathbb{R}} n(t,ct,y)dy \le \mu$. This proves (21) since μ may be taken arbitrarily small. □

To conclude the above proof, we require the control (50).

Lemma 5.1. *Suppose r, K and n_0 satisfy the assumptions of Theorem 2.11. Let n be any global nonnegative solution of* (1). *Then there exists $N_\infty > 0$ such that*

$$\int_{\mathbb{R}} n(t,x,y)dy \leq N_\infty. \tag{49}$$

Additionally, for every $\alpha > 0$ there exist $C', \kappa > 0$ such that

$$n(t,x,y) \leq C' e^{-\kappa|y-Bx|} + e^{-\alpha t} u(t,x) p(t,y), \tag{50}$$

where u and p respectively solve (42)*-*(43).

Proof of Lemma 5.1. The proof, very similar to that of [1, Lemma 2.4], is included here for the sake of completeness. The first assertion is straightforward. If we define the mass $N(t,x) := \int_{\mathbb{R}} n(t,x,y)dy$, an integration of (1) along the y variable provides the inequality

$$\partial_t N - \partial_{xx} N \leq N(r_{max} - k_- N).$$

Since $N_0(x) \leq ||u_0||_\infty(\sigma_+ - \sigma_-)$, it follows from the maximum principle that the mass is uniformly bounded :

$$N(t,x) \leq N_\infty := \max\left(||u_0||_\infty(\sigma_+ - \sigma_-), \frac{r_{max}}{k_-} \right),$$

which proves (49).

Let us now turn to the second assertion. Fix $R > 0$ large enough such that $\tilde{r}(Y) \leq -\alpha$ whenever $|Y| \geq R$. Set

$$\Omega_R := \left\{ (x,y) \in \mathbb{R}^2 \mid |Y| = \frac{|y - Bx|}{\sqrt{1+B^2}} < R \right\}.$$

Let us prove that n is uniformly bounded on Ω_R. In view of Assumption 2.1, there exists $M > 0$ such that for all $(x,y) \in \Omega_{R+1}$, there holds

$$\left| r(y - Bx) - \int_{\mathbb{R}} K(t,x,y,y') n(t,x,y') dy' \right| \leq \|r\|_{L^\infty(\Omega_R)} + k_+ N_\infty =: M.$$

As a consequence, n is the solution of a linear parabolic problem with bounded coefficients on $\overline{\Omega_{R+1}}$ (the nonlocal term being treated as a function of (t,x,y)), which allows us to apply the parabolic Harnack inequality (see [22] for instance). Fix any $\tau > 0$. There exists $C_H = C_H(\tau, R) > 0$ such that for all $t > 0$ and $\overline{x} \in \mathbb{R}$:

$$\max_{(x,y) \in B_R(\overline{x})} n(t,x,y) \leq C_H \min_{(x,y) \in B_R(\overline{x})} n(t+\tau, x, y),$$

where $B_R(\overline{x}) \subset \Omega_{R+1}$ denotes the closed ball of radius R of center $(\overline{x}, B\overline{x})$. This yields

$$\max_{(x,y) \in B_R(\overline{x})} n(t,x,y) \leq \frac{C_H}{2R} \int_{\mathbb{R}} n(t+\tau, \overline{x}, y) dy = \frac{C_H}{2R} N(t+\tau, \overline{x}) \leq \frac{C_H N_\infty}{2R}.$$

Seeing that C_H does not depend on $\overline{x}$, the population $n(t,x,y)$ is uniformly bounded by $\frac{C_H N_\infty}{2R}$ on $\mathbb{R}_+ \times \overline{\Omega_R}$.

To conclude, define, for any $\alpha > 0$,

$$\varphi(t,x,y) := C e^{-\kappa(|y-Bx| - R\sqrt{1+B^2})} + e^{-\alpha t} u(t,x) p(t,y),$$

where C, κ are positive constants, and u, p solve (42) and (43) respectively. Let us check that $n(t,x,y) \leq \varphi(t,x,y)$ on $\mathbb{R}_+ \times \mathbb{R}^2$. The inequality holds for $t = 0$ by (20). If we choose any $C \geq \frac{C_H N_\infty}{2R}$, there holds $n(t,x,y) \leq \varphi(t,x,y)$ on $\mathbb{R}_+ \times \overline{\Omega_R}$. Next, on the remaining region $(0, +\infty) \times \Omega_R^c$, we have $r(y - Bx) \leq -\alpha$, thus φ satisfies

$$\begin{aligned} \partial_t \varphi - \partial_{xx} \varphi - \partial_{yy} \varphi - r(y - Bx)\varphi &= \left(-\kappa^2(1+B^2) - r(y-Bx)\right) C e^{-\kappa(|y-Bx| - R\sqrt{1+B^2})} \\ &\quad + (-\alpha + ||u_0||_\infty - u(t,x) - r(y-Bx)) e^{-\alpha t} u(t,x) p(t,y), \\ &\geq \left(\alpha - \kappa^2(1+B^2)\right) C e^{\kappa R\sqrt{1+B^2}} e^{-\kappa|Y|}, \end{aligned}$$

which is nonnegative if we fix any $\kappa \leq \sqrt{\frac{\alpha}{1+B^2}}$. Since $n \geq 0$, it is a subsolution of the same operator. The maximum principle allows us to conclude that $n(t,x,y) \leq \varphi(t,x,y)$ on $(0,+\infty) \times \Omega_R^c$. We deduce (50) by setting $C' := Ce^{\kappa R\sqrt{1+B^2}}$. □

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