

Optimal strategies for mosquitoes replacement strategy: influence of the carrying capacity on spatial releases

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Abstract

This work is devoted to the mathematical study of an optimization problem regarding control strategies of mosquito population in a heterogeneous environment. Mosquitoes are well known to be vectors of diseases, but, in some cases, they have a reduced vector capacity when carrying the endosymbiotic bacterium *Wolbachia*. We consider a mathematical model of a replacement strategy, consisting in rearing and releasing *Wolbachia*-infected mosquitoes to replace the wild population. We investigate the question of optimizing the release protocol to have the most effective replacement when the environment is heterogeneous. In other words we focus on the question: where to release, given an inhomogeneous environment, in order to maximize the replacement across the domain. To do so, we consider a simple scalar model in which we assume that the carrying capacity is space dependent. Then, we investigate the existence of an optimal release profile and prove some interesting properties. In particular, neglecting the mobility of mosquitoes and under some assumptions on the biological parameters, we characterize the optimal releasing strategy for a short time horizon, and provide a way to reduce to a one-dimensional optimization problem the case of a long time horizon. Our theoretical results are illustrated with several numerical simulations.

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1 Introduction

Mosquitoes of the genus *Aedes* are responsible for the transmission of many diseases to humans. Among them, one may cite dengue, Zika, and chikungunya. One promising way to control the spread of such diseases is the use of the bacterium *Wolbachia* [10]. Indeed, it has been reported that *Wolbachia*-infected mosquitoes are less capable of transmitting such diseases [33, 7]. Moreover, this bacterium is characterized by, on the one hand, a vertical transmission from mother to offsprings and, on the other hand, producing a *cytoplasmic incompatibility*, which makes the mating

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of *Wolbachia*-infected males with wild females to be less fertile [29]. Taking advantage of these features, a population replacement strategy has been developed by rearing and releasing *Wolbachia*-infected mosquitoes [18]. This strategy is becoming very popular since it is potentially self-sustaining and has been successfully deployed in the field by the World Mosquito Program (see [16, 25] and <https://www.worldmosquitoprogram.org/>).

It has motivated several works including mathematical studies that have proved to be of great interest by proposing models that can be used to test and optimize different scenarios, see e.g. [19, 26, 3] and the review article [24]. In particular, mathematical control theory has been used to study the feasibility of the strategy, see e.g. [1, 9, 12]. A natural question is the optimisation of the release protocol. How to design the best release protocol, taking into account production constraints, to have the most efficient population replacement? Several mathematical works have addressed this issue. First, the optimisation of the temporal distribution of the releases, neglecting the spatial dependency, has been studied in e.g. [11, 6, 4, 2]. In particular, using a reduced model, the authors in [6] show that the best strategy is close to a bang-bang strategy where the maximum number of mosquitoes available is released at one time at the beginning or end (depending of the number of mosquitoes available) of the release period. Then, the study of the optimisation in space has been initiated in [5, 13, 23, 21] assuming that the space is homogeneous. In these works, the release is assumed to be done at initial time only and the question is to know how to spatially distribute the population to release in order to optimize the efficiency.

However, it is clear that for practical applications, environment cannot be considered as homogeneous and heterogeneity may have a strong influence on the dynamics of the replacement (see e.g. [22]). The aim of this paper is to propose a first mathematical study of an optimisation problem for a population replacement in a non-homogeneous landscape. More precisely, we consider the two-population model used in [6], see also [14, 15, 17, 32], where similar systems are considered, and assume that the carrying capacity of the environment is heterogeneous in space, modelling the fact that some areas are more favourable than others for the mosquito population. The system is first reduced in the spirit of [30], and the release is assumed to be pointwise in time: Indeed, it has been observed in [6, 3] that this is often the best strategy. Moreover, the spatial active motion of mosquitoes, usually modelled by a diffusion term, is neglected for the sake of simplicity. To fix the notation, let us present briefly the resulting optimisation problem considered in this article. Denote $p(t, x)$ the fraction of *Wolbachia*-infected population at time $t > 0$ and position $x \in \Omega$, with Ω being a bounded domain of $\mathbb{R}$ or $\mathbb{R}^2$. We seek to minimize the distance, at some given final time $T > 0$, between the proportion p and the total invasion constant steady state 1. Using the L^2 -norm, the problem reads

$$\min_{u_0 \in \mathcal{U}_{0,C,M}} \int_{\Omega} K(x)^2 (1 - p(T, x))^2 dx,$$

$$\mathcal{U}_{0,C,M} = \left\{ u_0 \in L^\infty(\Omega), \quad 0 \leq u_0 \leq M \text{ a.e.}, \quad \int_{\Omega} u_0(x) dx \leq C \right\}.$$

In this problem, K is the given carrying capacity depending on the space variable $x \in \Omega$, and u_0 is the initial release function which is positive, bounded by a constant M and such that its integral (which corresponds to the total quantity of mosquitoes released) is bounded by C . The relation between this release function and the proportion p is obtained by solving the differential equation

$$\partial_t p(t, x) = f(p(t, x)), \quad p(t = 0, x) = p_0(x) := G^{-1} \left(\frac{u_0(x)}{K(x)} \right).$$

In this system, f is a given bistable function and G is a given positive and increasing function. Their expressions will be given explicitly later. Solving this optimal control problem boils down

to investigate the question : what is the best initial distribution of the release in a heterogeneous environment in order to optimize the population replacement when active motion of mosquitoes is neglected?

Up to our knowledge, this work is the first one considering spatial heterogeneity in such an optimisation problem. We mention that neglecting active motion of mosquitoes allows us to reduce the problem to a dynamical system instead of considering a reaction-diffusion system, which greatly simplifies the study, and allows to derive precise results on the optimum. Although the study is simplified, it is very challenging to get a precise description of the optimal solution. This is a first step towards more sophisticated and more realistic future works where the active motion of mosquitoes will be considered. However, from a numerical point of view, we may draw some comparisons between numerical solutions of models with and without diffusion and check that, in certain conditions, they have a similar behaviour (see section 4.2).

The outline of the paper is the following. In the next section, we present the modelling assumptions and introduce the optimisation problem we are considering. Then, section 3 is devoted to the theoretical analysis of the optimisation problem and to the proof of our main results. Section 4 provides some numerical illustrations of our results. We also show numerical comparisons with a more realistic model with diffusion. Finally, some technical computations are provided in an appendix.

2 Mathematical model and optimal control problem

2.1 Simplified system

Let u denote a function accounting for the rate at which *Wolbachia*-infected mosquitoes are released in a given bounded domain, Ω , of $\mathbb{R}$ or $\mathbb{R}^2$. To model the dynamics of the replacement of a wild population of mosquitoes by *Wolbachia*-infected mosquitoes, the following system has been considered in e.g. [6]. In this system, the density of the *Wolbachia*-infected species is denoted n_2 , whereas the density of the wild population to be replaced is denoted n_1 . The system reads

$$\begin{cases} \partial_t n_1 - D\Delta n_1 = b_1 n_1 \left(1 - \frac{n_1 + n_2}{K(x)}\right) \left(1 - s_h \frac{n_2}{n_1 + n_2}\right) - d_1 n_1, \\ \partial_t n_2 - D\Delta n_2 = b_2 n_2 \left(1 - \frac{n_1 + n_2}{K(x)}\right) - d_2 n_2 + u, & t \in [0, T], \quad x \in \Omega \\ \partial_\nu n_1 = 0, \quad \partial_\nu n_2 = 0, & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

In this system, b_i , $i = 1, 2$, and d_i , $i = 1, 2$ represent the intrinsic birth and death rates of population i (i.e. not considering the term representing the population limitation due to the carrying capacity). In many insect species, among which we find mosquitoes, the birth rate is considerably higher than the death rate, therefore for biological reasons we will consider the constraint $d_i \leq b_i$. Also, we assume that the second population has a fitness disadvantage with respect to the first one, due to the infection with *Wolbachia*. This translates into imposing $b_2 \leq b_1$ and $d_1 \leq d_2$. The parameter s_h measures the *Wolbachia*-induced cytoplasmatic incompatibility (CI) of the second population over the first one. We have that $0 \leq s_h \leq 1$, when $s_h = 1$ there is a perfect CI, when $s_h = 0$ there is none. We assume that, at $t = 0$ there are no *Wolbachia*-infected individuals and that the wild population is at equilibrium, that is $n_2(0, x) = 0$ and $n_1(0, x) = K(x) \left(1 - \frac{d_1}{b_1}\right)$. The diffusion coefficient is denoted D and the carrying capacity K . In (2.1), $T > 0$ is the time horizon of the problem. In this work we consider that the carrying capacity K has Lipschitz spatial dependency:

$$0 < K(x) \in W^{1,\infty}(\Omega).$$

The goal we pursue is to find an optimal release function, u , such that at a given final time T the solution (n_1, n_2) to (2.1) is as close as possible of the n_2 -invasion steady state denoted $(0, n_2^*) = \left(0, K(x) \left(1 - \frac{d_2}{b_2}\right)\right)$. Choosing a least square distance, this leads us to introduce the following cost functional

$$J(u) = \frac{1}{2} \int_{\Omega} \left(n_1(T, x)^2 + [(n_2^* - n_2(T, x))_+]^2 \right) dx. \quad (2.2)$$

We consider some natural constraints on the number of available individuals to realize the experiments and also on the rate at which this individuals can be released. The set of admissible controls is therefore given by

$$\mathcal{U}_{T,C,M} = \left\{ u \in L^\infty([0, T] \times \Omega), \quad 0 \leq u \leq M \text{ a.e.}, \quad \int_0^T \int_{\Omega} u(t, x) dt dx \leq C \right\}. \quad (2.3)$$

This is inspired by the problem considered in [6]: here we assume that there is a maximum release rate M at each point in space and time and that there is a limited number of mosquitoes C that can be used during the whole intervention (up to time T). This is a simplified setting that is used in this work but that will be extended to more natural settings in future works. For instance, it is natural to consider a constraint on the initial number of mosquitoes available and a limited time flux of the mosquitoes which would correspond to the situation where the totality of the mosquitoes are not immediately available but will be progressively produced by a mosquito production facility. In that case, for extra realism, it is also possible to consider the fact that mosquitoes that are stocked continue to get older and that this should be taken into account in their life expectancy once they are released.

With the tools presented so far we can state the optimal control problem we deal with, namely:

$$\inf_{u \in \mathcal{U}_{T,C,M}} J(u), \quad (2.4)$$

where J is defined by (2.2) and $\mathcal{U}_{T,C,M}$ is defined by (2.3).

It has been proved in [30] and [6, Proposition 2.2] that when the fecundity rates are large, that is, if we assume that $b_1 = \frac{b_1^0}{\varepsilon}$ and $b_2 = \frac{b_2^0}{\varepsilon}$ and we let $\varepsilon \rightarrow 0$, system (2.1) may be reduced to one single equation on the proportion of *Wolbachia*-infected mosquitoes in the population. For the sake of completeness, we recall the formal computations leading to this system reduction. Let us assume that the fecundity rate is high, i.e. $b_1 = \frac{b_1^0}{\varepsilon}$ and $b_2 = \frac{b_2^0}{\varepsilon}$ with $\varepsilon \ll 1$. We introduce the total population $N = n_1 + n_2$ and the proportion of the species n_2 , $p = \frac{n_2}{n_1 + n_2}$. After straightforward computations, we obtain the system

$$\partial_t N - D \Delta N = N \left(\frac{1}{\varepsilon} \left(1 - \frac{N}{K} \right) (b_2^0 p + b_1^0 (1-p)(1-s_h p)) - d_2 p - d_1 (1-p) \right) + u, \quad (2.5)$$

$$\partial_t p - D \Delta p - 2D \frac{\nabla p \cdot \nabla N}{N} = p(1-p) \left(\frac{1}{\varepsilon} \left(1 - \frac{N}{K} \right) (b_2^0 - b_1^0 (1-s_h p)) + d_1 - d_2 \right) - \frac{1-p}{N} u. \quad (2.6)$$

We expect that, as $\varepsilon \rightarrow 0$, we have $N(t, x) = K(x)(1 + O(\varepsilon))$. Then we introduce an expansion of $N = N^\varepsilon(t, x)$ by letting

$$N = N^\varepsilon(t, x) = K(x) \left(1 - \varepsilon n^\varepsilon(t, x) + o(\varepsilon) \right).$$

Injecting into (2.5) and identifying the linear term in ε , we deduce the relation

$$n^\varepsilon(t, x) = \frac{d_2 p(t, x) + d_1 (1 - p(t, x)) - u/K(x)}{b_2^0 p(t, x) + b_1^0 (1 - p(t, x))(1 - s_h p(t, x))}.$$

Injecting this latter expression into the right hand side of (2.6), we obtain after straightforward computation that $p = p(t, x)$ solves the following scalar equation:

$$\begin{cases} \partial_t p - D\Delta p - 2D \frac{\nabla p \cdot \nabla K(x)}{K(x)} = f(p) + \frac{u(t, x)}{K(x)}g(p) - D \frac{\Delta K(x)}{K(x)}\psi(p), & t > 0, \quad x \in \Omega \\ p(0, x) = 0, & x \in \Omega, \\ \partial_\nu p(t, x) = 0, & t > 0, \quad x \in \partial\Omega; \end{cases} \quad (2.7)$$

where

$$f(p) = b_1^0 d_2 s_h \frac{p(1-p)(p-\theta)}{b_1^0(1-p)(1-s_h p) + b_2^0 p} \quad \text{and} \quad g(p) = \frac{b_1^0(1-p)(1-s_h p)}{b_1^0(1-p)(1-s_h p) + b_2^0 p}, \quad (2.8)$$

and

$$\psi(p) = \frac{p(1-p)(b_2^0 - b_1^0(1-s_h p))}{(1-p)(1-s_h p) + b_2^0 p}, \quad (2.9)$$

with

$$\theta = \frac{1}{s_h} \left(1 - \frac{d_1 b_2^0}{d_2 b_1^0} \right), \quad (2.10)$$

which is strictly comprised between 0 and 1 under the condition $1 - s_h < \frac{d_1 b_2^0}{d_2 b_1^0} < 1$, which will be assumed from now on. Moreover, the cost functional J reduces to

$$J^0(u) = \int_{\Omega} K(x)^2 (1 - p(T, x))^2 dx, \quad (2.11)$$

and we obtain an asymptotic version of Problem (2.4) reading

$$\inf_{u \in \mathcal{U}_{T,C,M}} \int_{\Omega} K(x)^2 (1 - p(T, x))^2 dx, \quad (2.12)$$

where p solves (2.7) and $\mathcal{U}_{T,C,M}$ is defined by (2.3). This model reduction allows us to study the problem in simpler terms, knowing that solutions of the simplified problem (2.12) will be asymptotically close to solutions of problem (2.4) in the sense of Gamma-convergence (see [13, 30] for details).

The case without considering a spatial variable has been studied in detail in [6]. Then the aim of this paper is to extend the former analysis to the case where space is considered by adding a global constraint on the whole domain and when the carrying capacity varies on it, which is usually the case in a natural environment.

2.2 Simplified optimal control problem for a single initial release

Despite all the simplifying assumptions made and the consequent the model reduction performed, problem (2.12) is still a very challenging one. To facilitate the study of this problem we will restrain ourselves to a further simplified setting. We first neglect the active motion of mosquitoes by considering that the diffusion coefficient $D = 0$. Then, we assume that the time distribution of the release is given by $u(t, x) = u_0(x)\delta_0(t)$. In other words, we consider that there is one single release, done at the initial time, and that the time it takes to do the release is negligible in comparison with the time window considered. Following the reasoning developed in [13], one can prove that equation (2.7) simplifies into

$$\begin{cases} \partial_t p(t, x) = f(p(t, x)), & t \in [0, T], \quad x \in \Omega \\ p(0^+, x) = G^{-1} \left(\frac{u_0(x)}{K(x)} \right), \end{cases} \quad (2.13)$$

where the function G is defined as the antiderivative vanishing at zero of $1/g$, i.e.

$$G(p) = \int_0^p \frac{d\nu}{g(\nu)}.$$

In this simplified setting we are looking for solutions to the optimal control problem

$$\boxed{\min_{u_0 \in \mathcal{U}_{0,C,U}} \int_{\Omega} K(x)^2 (1 - p(T, x))^2 dx,} \quad (\mathcal{P}_{u_0})$$

with the space of admissible controls being

$$\mathcal{U}_{0,C,M} = \left\{ u_0 \in L^\infty(\Omega), \quad 0 \leq u_0 \leq M \text{ a.e.,} \quad \int_{\Omega} u_0(x) dx \leq C \right\}. \quad (2.14)$$

Looking at (2.13) we see there is a one-to-one relation between the release carried at the initial time $u_0(x)$ and the initial data $p(0^+, x)$. We can reformulate problem $(\mathcal{P}_{u_0})$ in terms of this initial proportion by defining $p_0(x) := G^{-1} \left(\frac{u_0(x)}{K(x)} \right) = p(0^+, x)$ and considering it the new control variable of the problem. We find that Problem $(\mathcal{P}_{u_0})$ is equivalent to the following optimal control problem

$$\boxed{\min_{p_0 \in \mathcal{P}_{0,C,U}} \int_{\Omega} K(x)^2 (1 - p(T, x))^2 dx} \quad (\mathcal{P}_{p_0})$$

where p_0 is the initial data of the differential equation in (2.13) and the space of admissible controls is

$$\mathcal{P}_{0,C,M} = \left\{ p_0 \in L^\infty(\Omega), \quad 0 \leq p_0 \leq G^{-1} \left(\frac{M}{K(x)} \right) \text{ a.e.,} \quad \int_{\Omega} K(x) G(p_0(x)) dx \leq C \right\}. \quad (2.15)$$

Since g is decreasing, we have that G is convex. However, unless we add some restrictive assumptions on f , the cost functional is not necessarily convex, but is clearly continuous.

2.3 Main results

We summarize here in simple terms the main results contained in this work. These results will be detailed later in sections 3.4 and 3.5. To present our result, we introduce a so-called switch function, defined, for $T > 0$ by

$$w_T(p_0) := -g(p_0)(1 - p(T, x)) \exp \left(\int_0^T f'(p(s, x)) ds \right),$$

where we recall that $p(T)$ is the solution at time T of equation (2.13). Under certain hypothesis (which we assume and detail below) depending on the value of the final time considered, our results can be divided into two cases. The value separating the two regimes, that we call T_0 , can be determined as a function of the values of the biological parameters of the problem, see (3.25).

Our first result concerns the case $T \leq T_0$. This theorem is a simplified version of Theorem 3.6, which fully characterizes the solution of problem $(\mathcal{P}_{p_0})$ in this regime.

Theorem A ($T \leq T_0$) *Assume $T \leq T_0$ and $0 < C < M|\Omega|$. Then, there exists a unique $p_0^* \in \mathcal{P}_{0,C,M}$, that solves problem $(\mathcal{P}_{p_0})$. It is given by*

$$p_0^*(x) = \begin{cases} 0 & \text{if } -\frac{\lambda^*}{K(x)} \leq w_T(0), \\ p_M(x) := G^{-1} \left(\frac{M}{K(x)} \right) & \text{if } -\frac{\lambda^*}{K(x)} \geq w_T(p_M(x)), \\ w_T^{-1} \left(-\frac{\lambda^*}{K(x)} \right) & \text{if } -\frac{\lambda^*}{K(x)} \in (w_T(0), w_T(p_M(x))), \end{cases}$$

where $w_T(\cdot)$ as defined above is monotonically increasing and λ^* is any non-negative value such that its associated $p_0^*(x)$ satisfies $\int_{\Omega} K(x)G(p_0^*(x)) dx = C$.

By a monotonicity argument, one can prove that such a λ^* does exist and can be computed numerically. When it is not unique, the different values yield to the same optimum p_0^* . A straightforward consequence of this result is that, unless u_0^* saturates its L^∞ constraints, there is a positive correlation between $K(\cdot)$ and $u_0^*(\cdot)$. In other words, releases should be more intense where there are more mosquitoes to begin with. The reasoning goes as follows: for a given λ^* , $-\lambda^*/K(\cdot)$ increases in the direction that $K(\cdot)$ increases. Provided that $0 < u_0^* < M$, we have that $p_0^*(\cdot) = w_T^{-1}(\cdot)$, and since $w_T(\cdot)$ is monotonically increasing, so is $p_0^*(\cdot)$. Finally, since $u_0^*(\cdot) = K(\cdot)G(p_0^*(\cdot))$, with $G(\cdot)$ another strictly increasing function, the result follows. When we allow u_0^* to saturate the constraint $0 \leq u_0^* \leq M$, the correct conclusion is that u_0^* is non-decreasing where K is increasing. Note that this does not hold true for p_0^* , since p_M decreases when K increases. This will be further developed in Section 3.4.

Our second main result concerns the complementary case, $T > T_0$. It is given in Theorem 3.8 which may be summarized as:

Theorem B ($T > T_0$) *Assume $T > T_0$ and $0 < C < M|\Omega|$. Under appropriate conditions on the birth and death rates of mosquitoes, $w_T(\cdot)$ as defined above is unimodal, that is, first decreasing and then increasing, and any $p_0^* \in \mathcal{P}_{0,C,M}$ that solves problem $(\mathcal{P}_{p_0})$, if it exists, can be described as follows: Defining*

$$\tilde{p}_\lambda(x) = \begin{cases} 0 & \text{if } -\frac{\lambda}{K(x)} < \min_{p_0 \in (0, p_M(x))} w_T(p_0), \\ p_M(x) & \text{if } -\frac{\lambda}{K(x)} \geq w_T(p_M(x)), \\ w_T^{-1}\left(-\frac{\lambda}{K(x)}\right) & \text{if } -\frac{\lambda}{K(x)} \in \left[\min_{p_0 \in (0, p_M(x))} w_T(p_0), w_T(p_M(x))\right), \end{cases}$$

there exist $\lambda_1 \geq \lambda_0 \geq 0$ and at least one $\lambda^* \in [\lambda_0, \lambda_1]$ such that either $p_0^*(x) = \tilde{p}_{\lambda^*}(x)$ for all $x \in \Omega$, or, under some circumstances, the following holds:

- For each $\lambda \in [\lambda_0, \lambda_1]$, we introduce χ_λ^* , a characteristic function solving a new problem (stated in $(\mathcal{P}_{\tilde{\Omega}_\lambda})$) defined in a certain subdomain, $\tilde{\Omega}_\lambda \subset \Omega$, given by the parameters of the problem (see (3.32)). Then, there exists at least one $\lambda^* \in [\lambda_0, \lambda_1]$, such that p_0^* can be described as

$$p_0^*(x) = \begin{cases} \tilde{p}_{\lambda^*}(x), & \text{if } x \in \Omega \setminus \tilde{\Omega}_{\lambda^*}, \\ \tilde{p}_{\lambda^*}(x)\chi_{\lambda^*}^*, & \text{if } x \in \tilde{\Omega}_{\lambda^*}, \end{cases}$$

- Considering $p_0^*(x)$ as the initial condition of $p(\cdot, x)$, λ^* is a solution of the one-dimensional problem

$$\min_{\lambda \in [\lambda_0, \lambda_1]} \int_{\Omega} K(x)^2 (1 - p(T, x))^2 dx.$$

Depending on the values of the biological parameters, this result allows us to, either solve problem $(\mathcal{P}_{p_0})$ directly, or reduce it to a one dimensional problem on λ that can be solved numerically.

We remark that although we do not establish existence for a general carrying capacity function K , we were able to prove existence of a minimizer for certain specific forms of the function K , i.e. when K is constant (see Corollary 3.9) or, in one dimension, when K is piecewise constant (see Appendix B). Notice that since in practice K is estimated from field measures, which are done in a finite number of points in space, it is pertinent to consider the case where K is piecewise constant.

Exploiting these results, we implement an ad hoc numerical algorithm to visualize optimal solutions. Using the numerical software GEKKO (see [8]), we also compare these results with the numerical results obtained for the model with diffusion (2.7). Numerical results are displayed in section 4.

3 Analysis of the optimal problem

3.1 Preliminaries

Before analysing the problem in depth, we can obtain easily the following lemmas that will be useful later for the characterization of the solutions. First, we observe that the constraint of the problem is saturated and that the case $M|\Omega| \leq C$ is trivial:

Lemma 3.1 *If $u_0^* = K(x)G(p_0^*)$ is an optimal solution of the optimal control problem $(\mathcal{P}_{u_0})$, then $\int_{\Omega} u_0^*(x) dx = C$, or, equivalently, $\int_{\Omega} K(x)G(p_0^*(x)) dx = C$. Also, if $M|\Omega| \leq C$, the optimal solution is given by $u_0^* = M$ or equivalently $p_0^* = p_M := G^{-1}(\frac{M}{K})$.*

Proof. The first point is a trivial consequence of the fact that, on the one hand, G is increasing therefore so is G^{-1} , and on the other hand, the solutions of (2.13) are ordered, that is if $p_1(0^+, x) \leq p_2(0^+, x)$ then $p_1(\cdot, x) \leq p_2(\cdot, x)$.

Using also the monotonicity of the solutions of (2.13) with respect to their initial data, we get the second point. $\square$

Remark 3.2 We recall that, problems $(\mathcal{P}_{p_0})$ and $(\mathcal{P}_{u_0})$ are equivalent. Indeed, since G is continuous and strictly increasing, we have the simple one-to-one relation $u_0^*(x) := K(x)G(p_0^*(x))$.

3.2 Optimality conditions

As a consequence of previous lemma, we will always assume, from now on, that $M|\Omega| > C$. The following Lemma provides a description of the optimal solution using the optimality conditions:

Lemma 3.3 *Let us define the switch function w_T by*

$$w_T(p_0) = -g(p_0)(1 - p(T, x)) \exp \left(\int_0^T f'(p(s, x)) ds \right) < 0. \quad (3.16)$$

Let u_0^ be any optimal solution of the optimal control problem $(\mathcal{P}_{u_0})$. Then, there exists $\lambda^* \geq 0$ such that the optimal solution u_0^* verifies the first order conditions:*

- on $\{u_0^* = M\} = \left\{ p_0^* = p_M := G^{-1} \left(\frac{M}{K(x)} \right) \right\}$, we have $w_T \leq -\frac{\lambda^*}{K(x)}$,
- on $\{u_0^* = 0\} = \{p_0^* = 0\}$, we have $w_T \geq -\frac{\lambda^*}{K(x)}$,
- on $\{0 < u_0^* < M\} = \{0 < p_0^* < p_M\}$, we have $w_T = -\frac{\lambda^*}{K(x)}$ and each minimum should verify the second order condition

$$\frac{\partial w_T}{\partial p_0} \geq 0.$$

Remark 3.4 *It will be useful in the following to notice that in (3.16), the switch function depends on x only through the initial condition $p_0(x)$.*

Proof. We verify the first and second order optimality conditions.

• **First order optimality condition**

Let us introduce the Lagrangian

$$\mathcal{L}(p_0) = \frac{1}{2} \int_{\Omega} K(x)^2 (1 - p(T, x))^2 dx + \lambda \left(\int_{\Omega} K(x) G(p_0) dx - C \right),$$

for some $\lambda \in \mathbb{R}^+$. To compute its derivative, we introduce the linearized system

$$\partial_t \delta p = f'(p) \delta p, \quad \delta p(0^+, x) = h, \quad (3.17)$$

and the adjoint state

$$-\partial_t q = f'(p) q, \quad q(T, x) = -K(x)^2 (1 - p(T, x)) < 0. \quad (3.18)$$

In particular, from (3.17) we deduce

$$\frac{\partial p(T, x)}{\partial p_0(x)} = \exp \left(\int_0^T f'(p(s, x)) ds \right). \quad (3.19)$$

Then, to verify the first order optimality condition, we compute

$$d\mathcal{L}(p_0) \cdot h = - \int_{\Omega} K(x)^2 (1 - p(T, x)) \delta p(T, x) dx + \lambda \left(\int_{\Omega} K(x) G'(p_0) h dx \right).$$

Using (3.17) and (3.18), we deduce

$$0 = \int_0^T \int_{\Omega} \partial_t (\delta p q) dx dt = \int_{\Omega} \delta p(T, x) q(T, x) dx - \int_{\Omega} h q(0, x) dx.$$

Therefore,

$$d\mathcal{L}(p_0) \cdot h = \int_{\Omega} h (q(0, x) + \lambda K(x) G'(p_0)) dx = \int_{\Omega} \frac{K(x)}{g(p_0)} h \left(\frac{1}{K(x)} g(p_0) q(0, x) + \lambda \right) dx, \quad (3.20)$$

where we have used the fact that $G' = \frac{1}{g}$. Then, solving (3.18) we get

$$q(0, x) = -K(x)^2 (1 - p(T, x)) \exp \left(\int_0^T f'(p(s, x)) ds \right).$$

Injecting into (3.20), we obtain

$$d\mathcal{L}(p_0) \cdot h = \int_{\Omega} \frac{K(x)^2}{g(p_0(x))} h(x) \left(w_T(p_0(x)) + \frac{\lambda}{K(x)} \right) dx. \quad (3.21)$$

Then, noting that the function g is positive, thanks to a classical application of the Pontryagin Maximum Principle (PMP) [31], we obtain that there exists $\lambda^* > 0$ such that, on the set $\{u_0^* = 0\}$, we have $w_T + \frac{\lambda^*}{K(x)} \geq 0$, on the set $\{u_0^* = M\}$, we have $w_T + \frac{\lambda^*}{K(x)} \leq 0$, and on the set $\{0 < u_0^* < M\}$, we have $w_T + \frac{\lambda^*}{K(x)} = 0$.

- **Second order optimality condition**

We compute the second order derivative of the Lagrangian from the expression (3.21),

$$d^2\mathcal{L}(p_0) \cdot h \cdot h = \int_{\Omega} \frac{K(x)^2 g'(p_0)}{g(p_0)^2} h^2 \left(w_T + \frac{\lambda}{K(x)} \right) dx + \int_{\Omega} \frac{K(x)^2}{g(p_0)} h^2 \frac{\partial w_T}{\partial p_0} dx.$$

Then, on the set $\{0 < u_0^* < M\} = \{0 < p_0^* < p_M\}$, we have that

$$d^2\mathcal{L}(p_0) \cdot h \cdot h = \int_{\Omega} \frac{K(x)^2}{g(p_0)} h^2 \frac{\partial w_T}{\partial p_0} dx.$$

At the minimum it must hold that for every h , we have $d^2\mathcal{L}(p_0) \cdot h \cdot h \geq 0$. Since g is positive, the result follows. $\square$

3.3 Study of the switch function

We devote this section to the study of the switch function $w_T(p_0)$ defined in (3.16). This function depends on T and the initial condition $p_0(x)$. Nevertheless, in this section, we are only interested in its behaviour as a function of the initial condition. Therefore, in what follows, we fix a $x \in \Omega$ and we consider T as a parameter. To simplify the notation, we just write $w(p_0)$. Thus,

$$w(p_0) = -g(p_0)(1 - p(T)) \exp \left(\int_0^T f'(p(s)) ds \right)$$

We now present a Lemma on the monotonicity of w that will play a crucial role in the characterization of the solutions of problem $(\mathcal{P}_{p_0})$, yet, before stating it, we require some additional preliminaries and notations. Since we assumed $b_2^0 \leq b_1^0$ and $d_1 \leq d_2$, one can prove (see Appendix A) that f'' admits a unique zero θ_2 in $(0, 1)$. Additionally, for any $p \in [0, 1]$, we have $f''(p) > 0$ if and only if $p < \theta_2$. Setting $\bar{\theta} := \max(\theta, \theta_2)$, we now introduce the following function

$$p_0 \mapsto A(p_0) := \frac{g'(p_0)}{g(p_0)} - \frac{1}{1 - p(T)} e^{\int_0^T f'(p(s)) ds} + \int_0^T f''(p(s)) e^{\int_0^s f'(p(\sigma)) d\sigma} ds. \quad (3.22)$$

and the following hypothesis on it

$$\text{Function } p_0 \mapsto A(p_0) \text{ changes sign at most once in } (0, \bar{\theta}). \quad (\mathcal{H})$$

We investigate for which parameters $(\mathcal{H})$ is true in Appendix A.

Lemma 3.5 *Assume $(\mathcal{H})$ holds. There exists $T_0 > 0$ such that, if $T \leq T_0$, then $\frac{\partial w}{\partial p_0}(p_0) > 0$ for any $p_0 \in (0, 1)$. If $T > T_0$, there exists one single $p_0^T \in (0, \bar{\theta})$ such that*

$$\frac{\partial w}{\partial p_0}(p_0) < 0, \quad \forall p_0 \in (0, p_0^T) \quad \text{and} \quad \frac{\partial w}{\partial p_0}(p_0) > 0, \quad \forall p_0 \in (p_0^T, 1). \quad (3.23)$$

Let us notice that we have an explicit expression of T_0 which is given in (3.25) in the proof below.

Proof. We first look at the sign of $\frac{\partial w}{\partial p_0}$ at $p_0 = 0$ to derive the value of T_0 .

Sign of $\frac{\partial w}{\partial p_0}$ at $p_0 = 0$. Recalling (3.19), asimple computation yields

$$\begin{aligned} \frac{\partial w}{\partial p_0} &= -g'(p_0)(1-p(T))e^{\int_0^T f'(p(s))ds} + g(p_0)e^{2\int_0^T f'(p(s))ds} \\ &\quad - g(p_0)(1-p(T))e^{\int_0^T f'(p(s))ds} \int_0^T f''(p(s))e^{\int_0^s f'(p(\sigma))d\sigma} ds, \end{aligned} \quad (3.24)$$

As a result, we obtain

$$\begin{aligned} \left. \frac{\partial w}{\partial p_0} \right|_{p_0=0} &= -g'(0)e^{Tf'(0)} + g(0)e^{2Tf'(0)} - g(0)e^{Tf'(0)} \int_0^T f''(p(s))e^{s f'(0)} ds \\ &= -e^{Tf'(0)} \left[g'(0) - g(0)e^{Tf'(0)} + g(0) \frac{f''(0)}{f'(0)} (e^{Tf'(0)} - 1) \right]. \end{aligned}$$

Let us recall that $g(0) > 0 > g'(0)$ and $f'(0) < 0 < f''(0)$. From the above expression we deduce that

$$\begin{aligned} \left. \frac{\partial w}{\partial p_0} \right|_{p_0=0} &\geq 0 \\ \Leftrightarrow g'(0) - g(0)e^{Tf'(0)} + g(0) \frac{f''(0)}{f'(0)} (e^{Tf'(0)} - 1) &\leq 0 \\ \Leftrightarrow e^{Tf'(0)} g(0) \left(\frac{f''(0)}{f'(0)} - 1 \right) &\leq g(0) \frac{f''(0)}{f'(0)} - g'(0) \\ \Leftrightarrow e^{Tf'(0)} &\geq \frac{f''(0)g(0) - f'(0)g'(0)}{g(0)(f''(0) - f'(0))} \\ \Leftrightarrow T &\leq \frac{1}{f'(0)} \ln \left(\frac{f''(0)g(0) - f'(0)g'(0)}{g(0)(f''(0) - f'(0))} \right) =: T_0. \end{aligned} \quad (3.25)$$

One can check that the value of T_0 is always well defined and positive. Indeed, the argument of the logarithm is positive since

$$f''(0)g(0) - f'(0)g'(0) = \frac{b_2^0}{(b_1^0)^2} (2d_1 b_1^0 s_h + d_2 b_1^0 - b_2^0 d_1) > \frac{b_2^0}{(b_1^0)^2} 2d_1 b_1^0 s_h > 0.$$

We recall that $d_2 b_1^0 > b_2^0 d_1$ by the biological assumptions done so far. On the other hand, the argument of the logarithm is also smaller than one, since $g(0) = 1$ and $f''(0) - f'(0)g'(0) < f''(0) - f'(0)$. Consequently, if $T > T_0$, then $\frac{\partial w}{\partial p_0} < 0$ in a neighborhood of $p_0 = 0$.

Note that we can rewrite expression (3.24) using $A(p_0)$ as defined in equation (3.22),

$$\begin{aligned} \frac{\partial w}{\partial p_0} &= -g(p_0)(1-p(T))e^{\int_0^T f'(p(s))ds} \left[\frac{g'(p_0)}{g(p_0)} - \frac{1}{1-p(T)} e^{\int_0^T f'(p(s))ds} + \int_0^T f''(p(s))e^{\int_0^s f'(p(\sigma))d\sigma} ds \right] \\ &= w(p_0)A(p_0), \end{aligned} \quad (3.26)$$

Recall that $w(p_0) < 0$ and therefore $\frac{\partial w}{\partial p_0}$ changes signs as many times, and at the same points, as $A(p_0)$.

Fix $T > 0$ and set $\bar{\theta} := \max(\theta, \theta_2) \in (0, 1)$. Let us prove $\frac{\partial w}{\partial p_0} > 0$ for all $p_0 \in (\bar{\theta}, 1)$. From (3.26) and (3.16), it is enough to prove that $A = A(p_0)$ defined by (3.22) is negative on $(\bar{\theta}, 1)$. The first two terms of (3.22) are strictly negative for all $p_0 \in (0, 1)$. Therefore it is sufficient to prove that

$$\text{If } p_0 \in (\bar{\theta}, 1), \quad f''(p(t)) \leq 0 \text{ for all } t \in (0, T).$$

Let us recall that θ_2 is the unique zero of f'' in $(0, 1)$, and that $f'' < 0$ in $(\theta_2, 1)$ (see Proposition A.1). Let $p_0 > \bar{\theta}$. Then, since $p_0 > \theta$, one can quickly check that $t \mapsto p(t)$ is nondecreasing. Therefore, $p(t) \geq p_0 > \theta_2$, so that $f''(p(t)) \leq 0$ for all t . Thus $\frac{\partial w}{\partial p_0} > 0$ on $(\bar{\theta}, 1)$.

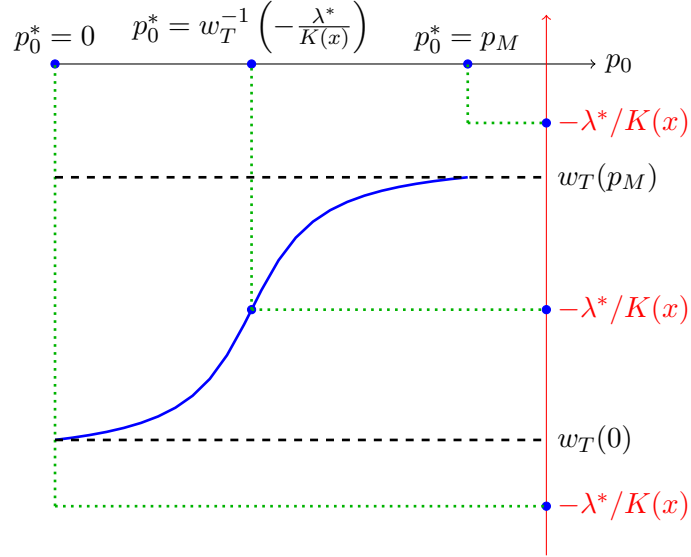


Figure 1: Typical shape of $p_0 \mapsto w_T(p_0)$, in the case $T \leq T_0$.

Conclusion. In conclusion, if $T \leq T_0$, we proved that $\frac{\partial w}{\partial p_0}|_{p_0=0}$ is positive and so is $\frac{\partial w}{\partial p_0}$ for all $p_0 \in (\bar{\theta}, 1)$. By Hypothesis $(\mathcal{H})$, $\frac{\partial w}{\partial p_0}$ changes sign at most once, by contradiction $\frac{\partial w}{\partial p_0}$ cannot change sign in $(0, \bar{\theta})$, and thus $\frac{\partial w}{\partial p_0} \geq 0$ for all $p_0 \in (0, 1)$. On the other hand, if $T > T_0$, $\frac{\partial w}{\partial p_0}|_{p_0=0} < 0$ and $\frac{\partial w}{\partial p_0} > 0$ for all $p_0 > \bar{\theta}$. Therefore w has at least one minimum. Again, by Hypothesis $(\mathcal{H})$, $\frac{\partial w}{\partial p_0}$ changes sign at most once, and thus the minimum, that we note p_0^T , must be unique. (3.23) follows straightforwardly. $\square$

3.4 The case $T \leq T_0$

First, we place ourselves in the case $T \leq T_0$. Let us introduce the following mappings defined on $\mathbb{R}^+$

$$\Lambda \mapsto \psi_{x,T}(\Lambda) := \begin{cases} 0 & \text{if } -\Lambda \leq w_T(0), \\ p_M(x) := G^{-1}\left(\frac{M}{K(x)}\right) & \text{if } -\Lambda \geq w_T(p_M(x)), \\ w_T^{-1}(-\Lambda) & \text{if } -\Lambda \in (w_T(0), w_T(p_M(x))). \end{cases} \quad (3.27)$$

and

$$\lambda \mapsto I(\lambda) := \int_{\Omega} K(x) G\left(\psi_{x,T}\left(\frac{\lambda}{K(x)}\right)\right) dx. \quad (3.28)$$

In the following Theorem, we state the main result for the case $T \leq T_0$. Theorem A presented in the beginning of this paper is a direct consequence of this result.

Theorem 3.6 *Assume $T \leq T_0$ and $0 < C < M|\Omega|$. Then there exists a unique $p_0^* \in \mathcal{P}_{0,C,M}$, that solves problem $(\mathcal{P}_{p_0})$. It is given by*

$$p_0^*(x) = \psi_{x,T}\left(\frac{\lambda^*}{K(x)}\right) \quad \text{for any } \lambda^* \text{ such that } I(\lambda^*) = C.$$

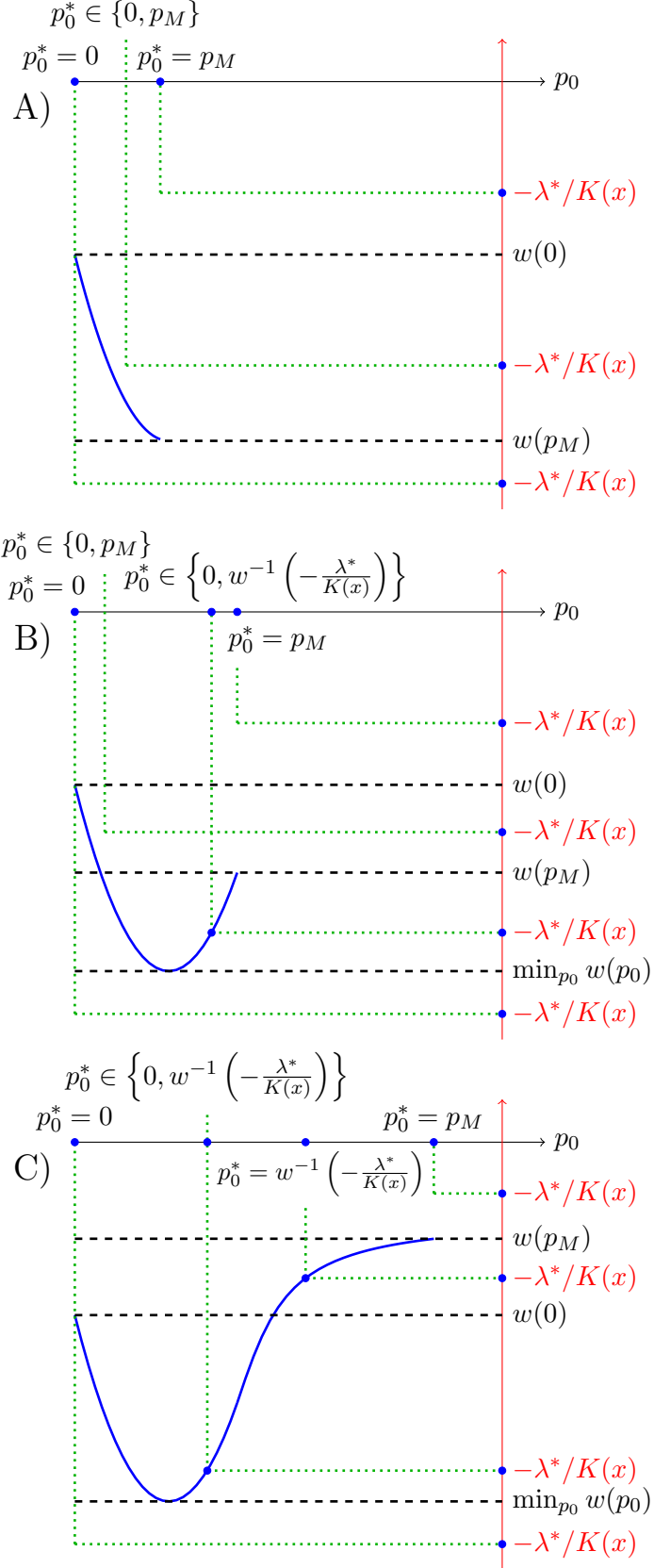


Figure 2: Schematic representation of w , as a function of p_0 in case $T > T_0$. As p_M increases (from top to bottom) the three diagrams, that we call A, B and C, show the three possible relative positions of $w(0)$, $w(p_M)$ and $\min_{p_0} w$.

Proof. Fix any $x \in \Omega$ and denote

$$w(p_0) := w_T(p_0(x)), \quad p_M = p_M(x) := G^{-1} \left(\frac{M}{K(x)} \right) > 0.$$

Let $\lambda^* \geq 0$ be any value given by Lemma 3.3 and assume $p_0^* = p_0^*(x)$ is any optimal control solving problem $(\mathcal{P}_{p_0})$. The optimality conditions given by Lemma 3.3 can be rewritten as:

- If $p_0^* = 0$, then $w(0) \geq -\frac{\lambda^*}{K(x)}$.
- If $p_0^* = p_M$, then $w(p_M) \leq -\frac{\lambda^*}{K(x)}$.
- If $0 < p_0^* < p_M$, then $w(p_0^*) = -\frac{\lambda^*}{K(x)}$.

Now, fix $T \leq T_0$. The above allows to compute the value $p_0^* = p_0^*(x)$ as follows. First, we look at the function $p_0 \rightarrow w(p_0)$ for $0 \leq p_0 \leq p_M$. Since, from Lemma 3.5, w is increasing, we have $w(0) \leq w(p_0) \leq w(p_M) < 0$ with $w(0) < w(p_M)$. Now, depending on the value λ^* , we distinguish three cases:

- If $-\frac{\lambda^*}{K(x)} \leq w(0)$, then necessarily $p_0^* = 0$.
- If $-\frac{\lambda^*}{K(x)} \geq w(p_M)$, then necessarily $p_0^* = p_M$.
- If $-\frac{\lambda^*}{K(x)} \in (w(0), w(p_M))$, then necessarily $p_0^* = w^{-1} \left(-\frac{\lambda^*}{K(x)} \right)$.

In other words, for any given $T \leq T_0$ and $x \in \Omega$ we have

$$p_0^*(x) = \psi_{x,T} \left(\frac{\lambda^*}{K(x)} \right) = \begin{cases} 0 & \text{if } -\frac{\lambda^*}{K(x)} \leq w_T(0), \\ p_M(x) & \text{if } -\frac{\lambda^*}{K(x)} \geq w_T(p_M(x)), \\ w_T^{-1} \left(-\frac{\lambda^*}{K(x)} \right) & \text{if } -\frac{\lambda^*}{K(x)} \in (w_T(0), w_T(p_M(x))). \end{cases} \quad (3.29)$$

As a consequence, if u_0^* is an optimal control, p_0^* must satisfy (3.29), meaning p_0^* is uniquely determined for a given λ^* . We claim that each value λ^* , given by Lemma (3.3), leads to the same function p_0^* , meaning p_0^* is uniquely determined. Consider $I(\lambda)$ as defined in (3.28). If p_0^* is optimal, then necessarily $I(\lambda^*) = C$, see Lemma 3.1. Note that the function $\psi_{x,T}$ is clearly continuous and nonincreasing, thus so is I . Also,

$$I(\lambda) = \int_{\Omega} K(x)G(0)dx = 0, \quad \text{if } \lambda \geq \lambda_{max} := -w_T(0) \min_x K(x),$$

$$I(\lambda) = \int_{\Omega} K(x)G(p_M(x))dx = M|\Omega|, \quad \text{if } \lambda \leq \lambda_{min} := -\max_x K(x)w_T(p_M(x)).$$

Since we assumed $0 < C < M|\Omega|$, we deduce that there exist $\lambda_{min} < \lambda_1^* \leq \lambda_2^* < \lambda_{max}$ such that

$$I(\lambda) = C \quad \forall \lambda \in [\lambda_1^*, \lambda_2^*].$$

Therefore, $\lambda^* \in [\lambda_1^*, \lambda_2^*]$. While λ^* is not uniquely determined, we claim that $\psi_{x,T}$ is constant on $[\lambda_1^*/K(x), \lambda_2^*/K(x)]$ for a.e. $x \in \Omega$. Assume by contradiction that there exists a set $S \subset \Omega$ with positive measure such that $\psi_{x,T}$ is nonconstant on $[\lambda_1^*/K(x), \lambda_2^*/K(x)]$ for all $x \in S$. This implies, since $\psi_{x,T}$ is nonincreasing,

$$\psi_{x,T} \left(\frac{\lambda_1^*}{K(x)} \right) > \psi_{x,T} \left(\frac{\lambda_2^*}{K(x)} \right), \quad \forall x \in S.$$

On the other hand, we also have

$$\psi_{x,T} \left(\frac{\lambda_1^*}{K(x)} \right) \geq \psi_{x,T} \left(\frac{\lambda_2^*}{K(x)} \right), \quad \forall x \in \Omega.$$

As a result, since G is increasing, we deduce that

$$\begin{aligned} I(\lambda_1^*) - I(\lambda_2^*) &= \int_{\Omega \setminus S} K(x) \left(G \left(\psi_{x,T} \left(\frac{\lambda_1^*}{K(x)} \right) \right) - G \left(\psi_{x,T} \left(\frac{\lambda_2^*}{K(x)} \right) \right) \right) dx \\ &\quad + \int_S K(x) \left(G \left(\psi_{x,T} \left(\frac{\lambda_1^*}{K(x)} \right) \right) - G \left(\psi_{x,T} \left(\frac{\lambda_2^*}{K(x)} \right) \right) \right) dx \\ &\geq \int_S K(x) \left(G \left(\psi_{x,T} \left(\frac{\lambda_1^*}{K(x)} \right) \right) - G \left(\psi_{x,T} \left(\frac{\lambda_2^*}{K(x)} \right) \right) \right) dx \\ &> 0 \end{aligned}$$

where the last inequality follows from the fact that $|S| > 0$ and $K(x) > \min_{\Omega} K > 0$. This contradicts the fact that $I(\lambda_1^*) = I(\lambda_2^*) = C$. Therefore, $\psi_{x,T}$ is constant on $[\lambda_1^*/K(x), \lambda_2^*/K(x)]$ for a.e. $x \in \Omega$. As a result, $p_0^*(x) = \psi_{x,T} \left(\frac{\lambda^*}{K(x)} \right)$ is uniquely determined, for any value $\lambda^* \in [\lambda_1^*, \lambda_2^*]$. $\square$

Remark 3.7 Notice that Theorem 3.6 implies that releases should be more important where the carrying capacity is high. Since λ^* is fixed, the argument of $\psi_{x,T}(\cdot)$, $\lambda^*/K(x)$, is smaller where $K(x)$ is higher. In case $-\frac{\lambda^*}{K(x)} \notin (w_T(0), w_T(p_M(x)))$, we have either $u_0(x) = 0$ or $u_0(x) = M$. On the other hand, in case $-\frac{\lambda^*}{K(x)} \in (w_T(0), w_T(p_M(x)))$, $\psi_{x,T} \left(\frac{\lambda^*}{K(x)} \right) = w_T^{-1} \left(-\frac{\lambda^*}{K(x)} \right)$. And since $w_T(\cdot)$ is monotonically increasing (see Figure 1), a bigger $K(x)$ implies a bigger argument (because of the minus sign), which implies a bigger $p_0^*(x)$. Since $u_0^*(x) = K(x)G(p_0^*(x))$, and G is also monotonically increasing, it follows that $u_0^*(x)$ must be non-decreasing when $K(x)$ increases in general, and strictly increasing with $K(x)$ whenever $u_0^*(x) \notin \{0, M\}$.

3.5 The case $T > T_0$

We study now the case $T > T_0$. In order to state the results in this case it will be useful to introduce some tools and notation. Let us introduce the following mappings

$$\Lambda \mapsto \psi_{x,T}^0(\Lambda) := \begin{cases} 0 & \text{if } -\Lambda \leq w_T(0), \\ p_M(x) & \text{if } -\Lambda > \max(w_T(0), w_T(p_M(x))), \\ w_T^{-1}(-\Lambda) & \text{if } -\Lambda \in (w_T(0), w_T(p_M(x))), \end{cases} \quad (3.30)$$

the third case only being defined if $w_T(0) < w_T(p_M(x))$, and

$$\Lambda \mapsto \psi_{x,T}^1(\Lambda) := \begin{cases} 0 & \text{if } -\Lambda < \min_{p_0 \in (0, p_M(x))} w_T(p_0), \\ p_M(x) & \text{if } -\Lambda \geq w_T(p_M(x)), \\ w_T^{-1}(-\Lambda) & \text{if } -\Lambda \in \left[\min_{p_0 \in (0, p_M(x))} w_T(p_0), w_T(p_M(x)) \right), \end{cases} \quad (3.31)$$

the third case only being defined if $\min_{p_0 \in (0, p_M(x))} w_T(p_0) < w_T(p_M(x))$. It is important to remark that $w^{-1} \left(-\frac{\lambda^*}{K(x)} \right)$ might not be uniquely defined, since the function is not injective on its entire domain.

Whenever there is an ambiguity it will be understood that the value of $w^{-1}\left(-\frac{\lambda^*}{K(x)}\right)$ we refer to, is the one on the increasing branch of $w(p_0)$ (the one satisfying $\frac{\partial w}{\partial p_0}(p_0) \geq 0$), since it is the only one satisfying the second order optimality conditions.

For a given value of $\lambda \geq 0$, let us introduce the set:

$$\tilde{\Omega}_\lambda := \left\{ x \in \Omega \mid \psi_{x,T}^0\left(\frac{\lambda}{K(x)}\right) \neq \psi_{x,T}^1\left(\frac{\lambda}{K(x)}\right) \right\}. \quad (3.32)$$

By definition, for all $x \in \Omega \setminus \tilde{\Omega}_\lambda$, $\psi_{x,T}^0\left(\frac{\lambda}{K(x)}\right) = \psi_{x,T}^1\left(\frac{\lambda}{K(x)}\right)$. In order to underline this, for $x \in \Omega \setminus \tilde{\Omega}_\lambda$ we will denote $\psi_{x,T}^\bullet\left(\frac{\lambda}{K(x)}\right) := \psi_{x,T}^0\left(\frac{\lambda}{K(x)}\right) = \psi_{x,T}^1\left(\frac{\lambda}{K(x)}\right)$. Note also that in case $\psi_{x,T}^0\left(\frac{\lambda}{K(x)}\right) \neq \psi_{x,T}^1\left(\frac{\lambda}{K(x)}\right)$, then $\psi_{x,T}^0\left(\frac{\lambda}{K(x)}\right) = 0$. Therefore, $\psi_{x,T}^0\left(\frac{\lambda}{K(x)}\right) = 0$ for all $x \in \tilde{\Omega}_\lambda$.

In the same spirit as in Theorem 3.6, the idea behind Theorem 3.8 is to write the solution in the form $p_0^*(x) = \psi_{x,T}^\bullet\left(\frac{\lambda}{K(x)}\right)$ for certain values of λ . Therefore, solutions in $\tilde{\Omega}_\lambda$ will be hard to characterize in general. In order to study solutions in this set we introduce a secondary problem, the solution of which will allow us to determine the solutions of problem $(\mathcal{P}_{p_0})$ in $\tilde{\Omega}_\lambda$. Assuming $|\tilde{\Omega}_\lambda| > 0$, we introduce the quantity

$$\tilde{C}_\lambda := C - \int_{\Omega \setminus \tilde{\Omega}_\lambda} K(x) G\left(\psi_{x,T}^\bullet\left(\frac{\lambda}{K(x)}\right)\right) dx.$$

Assuming $\tilde{C}_\lambda > 0$ we consider the following problem:

Secondary problem.

$$\boxed{\min_{\chi \in \mathcal{X}} \int_{\tilde{\Omega}_\lambda} K(x)^2 (1 - p(T, x))^2 \chi(x) + K(x)^2 (1 - \chi(x)) dx, \int_{\tilde{\Omega}_\lambda} K(x) G\left(\psi_{x,T}^1\left(\frac{\lambda^*}{K(x)}\right)\right) \chi(x) dx \leq \tilde{C}_\lambda,} \quad (\mathcal{P}_{\tilde{\Omega}_\lambda})$$

where the new control variable is $\chi_\lambda \in \mathcal{X}$ and

$$\mathcal{X} := \left\{ \chi_\lambda \in L^\infty(\tilde{\Omega}_\lambda) \mid 0 \leq \chi_\lambda \leq 1 \right\}.$$

Here $p(T, x)$ is assumed to have $\psi_{x,T}^1\left(\frac{\lambda}{K(x)}\right)$ as initial condition.

Finally, let us also define

$$I^0(\lambda) := \int_{\Omega} K(x) G\left(\psi_{x,T}^0\left(\frac{\lambda}{K(x)}\right)\right) dx \quad \text{and} \quad I^1(\lambda) := \int_{\Omega} K(x) G\left(\psi_{x,T}^1\left(\frac{\lambda}{K(x)}\right)\right) dx. \quad (3.33)$$

and

$$\lambda_0 := \min \lambda \quad \text{such that} \quad I^0(\lambda) = C, \quad (3.34)$$

$$\lambda_1 := \max \lambda \quad \text{such that} \quad I^1(\lambda) = C. \quad (3.35)$$

Note that as long as $0 < C < M|\Omega|$ these two quantities will always be well defined.

Theorem 3.8 *Assume $T > T_0$, $0 < C < M|\Omega|$ and $(\mathcal{H})$. Then, any solution of problem $(\mathcal{P}_{p_0})$, $p_0^* \in \mathcal{P}_{0,C,M}$, has to satisfy:*

$$p_0^*(x) = \psi_{x,T}^\bullet\left(\frac{\lambda^*}{K(x)}\right) \quad \text{for all } x \in \Omega \setminus \tilde{\Omega}_{\lambda^*}, \quad \text{and} \quad p_0^*(x) = \begin{cases} \psi_{x,T}^1\left(\frac{\lambda^*}{K(x)}\right), & \text{for } x \in \tilde{\Omega}_{\lambda^*} \text{ s.t. } \chi_{\lambda^*}^*(x) = 1, \\ \psi_{x,T}^0\left(\frac{\lambda^*}{K(x)}\right), & \text{for } x \in \tilde{\Omega}_{\lambda^*} \text{ s.t. } \chi_{\lambda^*}^*(x) = 0, \end{cases}$$

where $\psi_{x,T}^0\left(\frac{\lambda^*}{K(x)}\right) = 0$ for all $x \in \tilde{\Omega}_{\lambda^*}$, $\lambda^* \in [\lambda_0, \lambda_1]$ and $\chi_{\lambda^*}^*(x)$ is the solution to problem $(\mathcal{P}_{\tilde{\Omega}_{\lambda^*}})$ with $\lambda = \lambda^*$.

Furthermore, if $|\tilde{\Omega}_{\lambda_0}| = 0$, then $\lambda^* = \lambda_0$ and $p_0^*(x) = \psi_{x,T}^\bullet\left(\frac{\lambda_0}{K(x)}\right)$ for all $x \in \Omega$.

Proof.

In this proof we use the same notation as in the proof of Theorem 3.6. Let us fix any $x \in \Omega$ and let $\lambda^* \geq 0$ be any value given by Lemma 3.3. Let $p_0^* \in \mathcal{P}_{0,C,M}$ be the solution of problem $(\mathcal{P}_{p_0})$.

Using the first and second order optimality conditions (see Lemma (3.3)) we know that there exists a $\lambda^* \geq 0$ such that the optimal control p_0^* must satisfy:

- If $p_0^* = 0$ then $w(0) \geq -\frac{\lambda^*}{K(x)}$.
- If $p_0^* = p_M$ then $w(p_M) \leq -\frac{\lambda^*}{K(x)}$.
- If $0 < p_0^* < p_M$ then $w(p_0^*) = -\frac{\lambda^*}{K(x)}$ and $\frac{\partial w}{\partial p_0}(p_0^*) \geq 0$.

We fix $T > T_0$ and exploit these optimality conditions. Under hypothesis $(\mathcal{H})$, by Lemma 3.5, $w(p_0)$ is unimodal, that is, strictly decreasing until a certain $p_0^T \in (0, \bar{\theta})$ and then strictly increasing. Depending on the relative position of $w(0), w(p_M)$ and $\min w(p_0)$ we can have three different behaviours of the optimal control. We detail here as an example the case $w(0) \geq w(p_M) > \min w(p_0)$. Cases $w(0) > w(p_M) \geq \min w(p_0)$ and $w(p_M) > w(0) > \min w(p_0)$ can be deduced from this analysis and Figure 2.

Assume p_M is such that $w(0) \geq w(p_M) > \min w(p_0)$ (case B in Figure 2), then we distinguish four cases:

- If $-\frac{\lambda^*}{K(x)} \geq w(0)$, then necessarily $p_0^* = p_M$.
- If $w(0) \geq -\frac{\lambda^*}{K(x)} \geq w(p_M)$, then necessarily $p_0^* \in \{0, p_M\}$.
- If $w(p_M) \geq -\frac{\lambda^*}{K(x)} \geq \min w(p_0)$, then necessarily $p_0^* \in \left\{0, w^{-1}\left(-\frac{\lambda^*}{K(x)}\right)\right\}$.
- If $-\frac{\lambda^*}{K(x)} \leq \min w(p_0)$, then necessarily $p_0^* = 0$.

Two things remain to be studied: The values that λ^* can take, and, in case p_0^* is not uniquely determined, how to choose between the two options. In order to do this, let us consider mappings $\psi_{x,T}^0$ and $\psi_{x,T}^1$ as defined in (3.30) and (3.31) respectively. Note that these two mappings are always well defined and they give, respectively, the minimum and maximum values p_0^* can take when two values of p_0 satisfy the optimality conditions. For instance, if for a given value of λ^* , $p_0^* \in \{0, p_M\}$, then $\psi_{x,T}^0\left(\frac{\lambda^*}{K(x)}\right) = 0$ and $\psi_{x,T}^1\left(\frac{\lambda^*}{K(x)}\right) = p_M$. Remark also that whenever $\psi_{x,T}^0\left(\frac{\lambda}{K(x)}\right) \neq \psi_{x,T}^1\left(\frac{\lambda}{K(x)}\right)$, $\psi_{x,T}^0\left(\frac{\lambda}{K(x)}\right) = 0$.

Mappings $\psi_{x,T}^0\left(\frac{\lambda}{K(x)}\right), \psi_{x,T}^1\left(\frac{\lambda}{K(x)}\right)$ are non-increasing with respect to λ . Although in case $T > T_0$, $\psi_{x,T}^0$ and $\psi_{x,T}^1$ are only continuous in case C (see Figure 2), it still holds that for all $\lambda \geq 0$ we have that $I^0(\lambda) \leq I^1(\lambda)$, with $I^0(\lambda), I^1(\lambda)$, as defined in (3.33). Furthermore, if λ^* is any value given by Lemma 3.3 we have that

$$I^0(\lambda^*) \leq \int_{\Omega} K(x)G(p_0^*(x))dx \leq I^1(\lambda^*). \quad (3.36)$$

Using Lemma 3.1, (3.36) means that $I^0(\lambda^*) \leq C \leq I^1(\lambda^*)$. It follows that $\lambda^* \in [\lambda_0, \lambda_1]$.

Fixing now a $\lambda^* \in [\lambda_0, \lambda_1]$, let us consider the set $\tilde{\Omega}_{\lambda^*}$ as defined in (3.32). Note that if $|\tilde{\Omega}_{\lambda^*}| = 0$, we can conclude that $p_0^*(x) = \psi_{x,T}^\bullet \left(\frac{\lambda^*}{K(x)} \right)$ using the same arguments as in the proof of Theorem 3.6. In particular, if $|\tilde{\Omega}_{\lambda_0}| = 0$, since $p_0^*(x) = \psi_{x,T}^\bullet \left(\frac{\lambda_0}{K(x)} \right)$, $\psi_{x,T}^\bullet(\lambda)$ is non-increasing w.r.t λ and $\int_{\Omega} K(x)^2 (1 - p(T, x))^2 dx$ is decreasing w.r.t to the initial data of $p(T, x)$, we can conclude that $\lambda^* = \lambda_0$, proving the last statement of the Theorem.

For the rest of the proof we assume that $|\tilde{\Omega}_{\lambda^*}| > 0$. Note also that the optimal control p_0^* must be such that $0 < \tilde{C}_{\lambda^*} \leq C$. We consider problem $(\mathcal{P}_{\tilde{\Omega}_{\lambda^*}})$ with $\lambda = \lambda^*$. Observing that the criterion to optimize is affine with respect to χ_{λ^*} and that its differential at $\chi_{\lambda^*}^*$ is the linear mapping

$$L^\infty(\Omega) \ni h \mapsto \int_{\tilde{\Omega}_{\lambda^*}} h K(x) \left(K(x)p(T, x)(p(T, x) - 2) + \tilde{\lambda}G \left(\psi_{x,T}^1 \left(\frac{\lambda^*}{K(x)} \right) \right) \right) dx,$$

leads to introduce the switch function Φ for this problem, namely

$$\Phi(x) := K(x) \frac{p(T, x)(2 - p(T, x))}{G \left(\psi_{x,T}^1 \left(\frac{\lambda^*}{K(x)} \right) \right)}.$$

We infer from the so-called bathtub principle (see e.g. Section 1.14 of [20]) the existence of a unique real number $\tilde{\lambda}^*$ such that

$$\{\tilde{\lambda}^* > \Phi\} \subset \{\chi_{\lambda^*}^* = 0\} \subset \{\tilde{\lambda}^* \geq \Phi\}, \quad \{\tilde{\lambda}^* < \Phi\} \subset \{\chi_{\lambda^*}^* = 1\} \subset \{\tilde{\lambda}^* \leq \Phi\}$$

and furthermore, $\{0 < \chi_{\lambda^*}^* < 1\} \subset \{\tilde{\lambda}^* = \Phi\}$. Note that such inclusions must be understood up to a zero Lebesgue measure set.

Let us denote $D := \left\{ x \in \tilde{\Omega}_{\lambda^*} \mid 0 < \chi_{\lambda^*}^*(x) < 1 \right\}$. In case $|D| = 0$, the optimality conditions become necessary and sufficient. The solution can be written as

$$\chi_{\lambda^*}^*(x) = \begin{cases} 1 & , \text{ if } \tilde{\lambda}^* < \Phi, \\ 0 & , \text{ if } \tilde{\lambda}^* > \Phi. \end{cases}$$

In case $|D| > 0$, since the problem is linear in χ_{λ^*} we know that there exists a bang-bang solution. That is, a solution that only takes the values $\chi_{\lambda^*}^* = 0$ and $\chi_{\lambda^*}^* = 1$. This means that despite it may exist a solution with $0 < \chi_{\lambda^*}^*(x) < 1$ for $x \in D$, we can always construct a bang-bang alternative that performs just as good. Assuming $\tilde{\lambda}^* = \Phi$ in D (Φ is constant in D by definition), we introduce

$$\chi_{\lambda^*}^\alpha(x) = \begin{cases} 1, & \text{ if } \tilde{\lambda}^* < \Phi \text{ or } x \in D^\alpha, \\ 0, & \text{ if } \tilde{\lambda}^* > \Phi \text{ or } x \in D \setminus D^\alpha. \end{cases}$$

where D^α is any subset of D such that $|D^\alpha|/|D| = \alpha$, and $\alpha \in [0, 1]$. To compute the value of α we use one more time Lemma 3.1, concluding that, in this case, $\chi_{\lambda^*}^*(x) = \chi_{\lambda^*}^\alpha(x)$ for α such that $\int_{\tilde{\Omega}_{\lambda^*}} K(x)G \left(\psi_{x,T}^1 \left(\frac{\lambda^*}{K(x)} \right) \right) \chi_{\lambda^*}^\alpha(x) dx = \tilde{C}_{\lambda^*}$. Note how the solution to this secondary problem, $(\mathcal{P}_{\tilde{\Omega}_{\lambda^*}})$, sheds light on the primary problem, $(\mathcal{P}_{p_0})$, by allowing us to write the solution on $\tilde{\Omega}_{\lambda^*}$ as

$$p_0^*(x) = \begin{cases} \psi_{x,T}^1(\lambda^*), & \text{ on } \chi_{\lambda^*}^*(x) = 1, \\ \psi_{x,T}^0(\lambda^*), & \text{ on } \chi_{\lambda^*}^*(x) = 0, \end{cases}$$

concluding the proof. $\square$

The last result concerns the particular case where K is a constant.

Corollary 3.9 Assume $0 < C < M|\Omega|$ and $K(x) = K$ constant in all Ω . Then there exists a $p_0^* \in \mathcal{P}_{0,C,M}$ that solves problem $(\mathcal{P}_{p_0})$.

- If $T \leq T_0$, it is given by

$$p_0^*(x) = G^{-1} \left(\frac{C}{K|\Omega|} \right) \text{ for all } x \in \Omega.$$

- If $T > T_0$,

- If $w(0) < w \left(G^{-1} \left(\frac{C}{K|\Omega|} \right) \right)$, then

$$p_0^*(x) = G^{-1} \left(\frac{C}{K|\Omega|} \right) \text{ for all } x \in \Omega$$

- If $w(0) \geq w \left(G^{-1} \left(\frac{C}{K|\Omega|} \right) \right)$, then there exists at least one $\lambda^* \in [\lambda_0, \lambda_1]$ such that $p_0^*(x)$ can be written as

$$p_0^*(x) = \begin{cases} \psi_{x,T}^1 \left(\frac{\lambda^*}{K} \right), & \text{for } x \in D, \\ 0, & \text{for } x \in \Omega \setminus D, \end{cases}$$

where D can be any subdomain of Ω with size $|D| = \frac{C}{KG(\psi_{x,T}^1(\frac{\lambda^*}{K}))}$.

Proof.

The existence of a unique solution written as $p_0^*(x) = \psi_{x,T} \left(\frac{\lambda^*}{K(x)} \right)$ for the case $T \leq T_0$ can be easily adapted from the proof of Theorem 3.6. Since the constraint must be saturated, we have that

$$\int_{\Omega} K(x) G \left(\psi_{x,T} \left(\frac{\lambda^*}{K(x)} \right) \right) dx = K \int_{\Omega} G \left(\psi_{x,T} \left(\frac{\lambda^*}{K} \right) \right) dx = C,$$

but for K constant, w is constant w.r.t. x , and thus, so is $\psi_{x,T} \left(\frac{\lambda^*}{K} \right)$. Therefore,

$$K \int_{\Omega} G \left(\psi_{x,T} \left(\frac{\lambda^*}{K} \right) \right) dx = KG \left(\psi_{x,T} \left(\frac{\lambda^*}{K} \right) \right) |\Omega| = C.$$

Concluding that

$$p_0^*(x) = \psi_{x,T} \left(\frac{\lambda^*}{K} \right) = G^{-1} \left(\frac{C}{K|\Omega|} \right).$$

The case $T > T_0$ is greatly simplified in this setting. Since $\psi_{x,T} \left(\frac{\lambda^*}{K} \right)$ is constant w.r.t. x , either $|\tilde{\Omega}_{\lambda}| = 0$, or $\tilde{\Omega}_{\lambda} = \Omega$. Like in the general case, if $|\tilde{\Omega}_{\lambda_0}| = 0$ we have that $p_0^*(x) = \psi_{x,T}^{\bullet} \left(\frac{\lambda_0}{K} \right) = G^{-1} \left(\frac{C}{K|\Omega|} \right)$, and by the monotonicity of $\int_{\Omega} (1 - p(T, x))^2 dx$ w.r.t to the initial condition of $p(T, x)$ we can conclude. On the other hand, in this case, we can put the condition $|\tilde{\Omega}_{\lambda_0}| = 0$ in simpler terms. If $|\tilde{\Omega}_{\lambda_0}| = 0$, then $\psi_{x,T}^0 \left(\frac{\lambda_0}{K} \right) = \psi_{x,T}^1 \left(\frac{\lambda_0}{K} \right)$. Looking at the two functions, this happens if and only if $w(0) < -\frac{\lambda_0}{K}$ and we have that

$$w(0) < -\frac{\lambda_0}{K} = w(p_0^*(x)) = w \left(\psi_{x,T}^{\bullet} \left(\frac{\lambda_0}{K} \right) \right) = w \left(G^{-1} \left(\frac{C}{K|\Omega|} \right) \right).$$

Therefore,

$$|\tilde{\Omega}_{\lambda_0}| = 0 \text{ if and only if } w(0) < w \left(G^{-1} \left(\frac{C}{K|\Omega|} \right) \right).$$

In case $w(0) \geq w\left(G^{-1}\left(\frac{C}{K|\Omega|}\right)\right)$, we have that $|\tilde{\Omega}_{\lambda_0}| > 0$ and thus, $|\tilde{\Omega}_{\lambda^*}| > 0$. Furthermore, since K is constant, $\tilde{\Omega}_{\lambda^*} = \Omega$.

Fixing a value for λ^* , $p_0^*(x)$ can only take two values in Ω , $p_0^*(x) = 0$ or $p_0^*(x) = \psi_{x,T}^1\left(\frac{\lambda^*}{K}\right)$. Therefore, from Lemma 3.1 we can directly deduce the size of the domain where $p_0^*(x) = \psi_{x,T}^1\left(\frac{\lambda^*}{K}\right)$, that we denote D ,

$$\int_{\Omega} KG(p_0^*(x)) dx = \int_D KG\left(\psi_{x,T}^1\left(\frac{\lambda^*}{K}\right)\right) dx = |D|KG\left(\psi_{x,T}^1\left(\frac{\lambda^*}{K}\right)\right) = C.$$

Thus,

$$|D| = \frac{C}{KG(\psi_{x,T}^1(\frac{\lambda^*}{K}))}.$$

Therefore, the solution can be written as $p_0^*(x) = \psi_{x,T}^1\left(\frac{\lambda^*}{K}\right)$ a.e. on D , and $p_0^*(x) = 0$ elsewhere, with $|D| = \frac{C}{KG(\psi_{x,T}^1(\frac{\lambda^*}{K}))}$. □

4 Numerical Implementation of results

Thanks to Theorems 3.6 and 3.8 we can implement an algorithm for computing solutions to problems $(\mathcal{P}_{u_0})$ and $(\mathcal{P}_{p_0})$. In the case $T \leq T_0$, the computations will be a direct application of the results presented in Theorem 3.6, where solutions are unique up to a rearrangement. In the case $T > T_0$, Theorem 3.8 allows us to significantly simplify the problem by recasting it as a one-dimensional one when solutions cannot be found directly. Namely,

$$\boxed{\min_{\lambda \in [\lambda_0, \lambda_1]} \int_{\Omega} K(x)^2 (1 - p(T, x))^2 dx} \quad (\mathcal{Q})$$

where we assume that the initial condition for $p(T, x)$ is given by the optimal release strategy, $p_0^*(x)$, given by theorem 3.8, assuming that $\lambda = \lambda^*$. Following the lines of [13], we present the simulations in the context of mosquito population replacement using *Wolbachia*, although as discussed in Section 1, the case with diffusion is more relevant in this setting. The parameters used for the simulations are presented in Table 1.

4.1 1D simulations

The simplest setting in which we can study the problem is considering only one spatial dimension. We present two examples of the solutions obtained exploiting the results proven in this paper in a 1D setting. We consider the following functions representing the carrying capacity of the environment

$$K_S(x) = K_0 \left(1 - \frac{1}{2} \cos\left(\frac{2\pi x}{|\Omega|}\right)\right), \quad K_P(x) = \frac{3K_0}{2} \mathbb{1}_{[0, |\Omega|/2]} + \frac{K_0}{2} \mathbb{1}_{(|\Omega|/2, |\Omega|]}$$

With the parameters considered, the two functions have the same average carrying capacity, K_0 . That is, $\int_{\Omega} K_S(x) dx = \int_{\Omega} K_P(x) dx = K_0|\Omega|$, which is also the value we would obtain in case of an homogeneous carrying capacity equal to K_0 in all the domain. The domain considered is $\Omega = [0, |\Omega|]$. We chose these two functions in order to have a piece-wise constant function and a function that is non-constant in any positive measure interval for comparison.

Category	Parameter	Name	Value
Optimization	T	Final time	$\{1, 25\}$
	M	Maximal instantaneous release rate	250
	C	Amount of mosquitoes available	$\{30, 200\}$
	$ \Omega $	Domain size	1
	K_0	Average carrying capacity	100
Biology	b_1^0	Normalized wild birth rate	1
	b_2^0	Normalized infected birth rate	0.9
	d_1	Wild death rate	0.27
	d_2	<i>Wolbachia</i> infected death rate	0.3
	s_h	Cytoplasmatic incompatibility level	0.9

Table 1: Values of the parameters used in simulations. The values for the biological parameters have been taken from [6].

For the parameters in Table 1, $A(\cdot)$, as defined in (3.22), satisfies hypothesis $(\mathcal{H})$. The time when the function stops being increasing and starts being unimodal (decreasing, then increasing) is $T_0 \approx 3.51$, computed using formula (3.25). We choose to show the results for a time smaller than T_0 and a time bigger than T_0 , so both behaviours can be observed. The simulations have been performed using an ad hoc algorithm exploiting the results of Theorems 3.6 and 3.8.

In Figure 3 we can see the results for $K(\cdot) = K_S(\cdot)$. This choice models a scenario where mosquitoes are concentrated in the center of the domain studied and their concentration fades out as we move towards the boundaries. In the case $C = 30$ (left column), we observe how the optimal strategy flattens and widens as T increases. To understand this effect, we recall that in case $p_0(x) < \theta$, $p(t, x)$ is decreasing with respect to t . In other words, the *Wolbachia*-infected mosquitoes tend to be replaced by wild mosquitoes if they don't surpass a critical threshold. Furthermore, if $p_0(x) > \theta$, $p(t, x)$ is increasing with respect to t , and therefore *Wolbachia*-infected mosquitoes take over the population without further intervention. For the parameters considered we have $\theta \approx 0.21$ (The green dash-dotted line in Figure 3). According to our interpretation, since for T small, $p(T, x)$ is close to its initial condition, a value of $p_0(x)$ below the threshold does not impact greatly the final result and reaching a bigger initial proportion in places where $K(x)$ is higher is favoured. On the other hand, when T is big, $p(T, x)$ can be far from its initial condition. Hence, there is an incentive for $p_0^*(x)$ to be above θ , since the proportion of *Wolbachia*-infected mosquitoes in the parts of the domain above θ will naturally increase with time. This also explains why the end of the release interval is abrupt. If the release is not going to achieve the critical proportion, it is better not to release. This effect can be seen clearly in the bottom-left graph and should be more pronounced the bigger T is.

The changes between $T = 1$ and $T = 25$ in the case $C = 200$ are imperceptible. A possible explanation is that in this case $p_0^*(x) > \theta$ for all $x \in \Omega$ already in the first case. Therefore the proportion of *Wolbachia*-infected mosquitoes is going to increase with time everywhere. This case illustrates how $u_0^*(\cdot)$ is non-decreasing as $K(\cdot)$ increases, and how that is not necessarily the case of $p_0^*(\cdot)$, which decreases when $K(\cdot)$ increases if $p_0^*(\cdot) = p_M(\cdot)$. We recall that we have proven this monotonicity property for the case $T \leq T_0$ and in case $T > T_0$ but $|\hat{\Omega}_{\lambda_0}| = 0$, which is the case in the bottom-right graph ($C = 200$, $T = 25$). The only case where $|\hat{\Omega}_{\lambda_0}| > 0$ in Figure 3 is the one with $C = 30$, $T = 25$ (bottom-left graph).

In Figure 4 we show the results for the simulations with $K(\cdot) = K_P(\cdot)$. This figure represents a scenario with two patches of land with two very distinct conditions for mosquitoes, as it can be

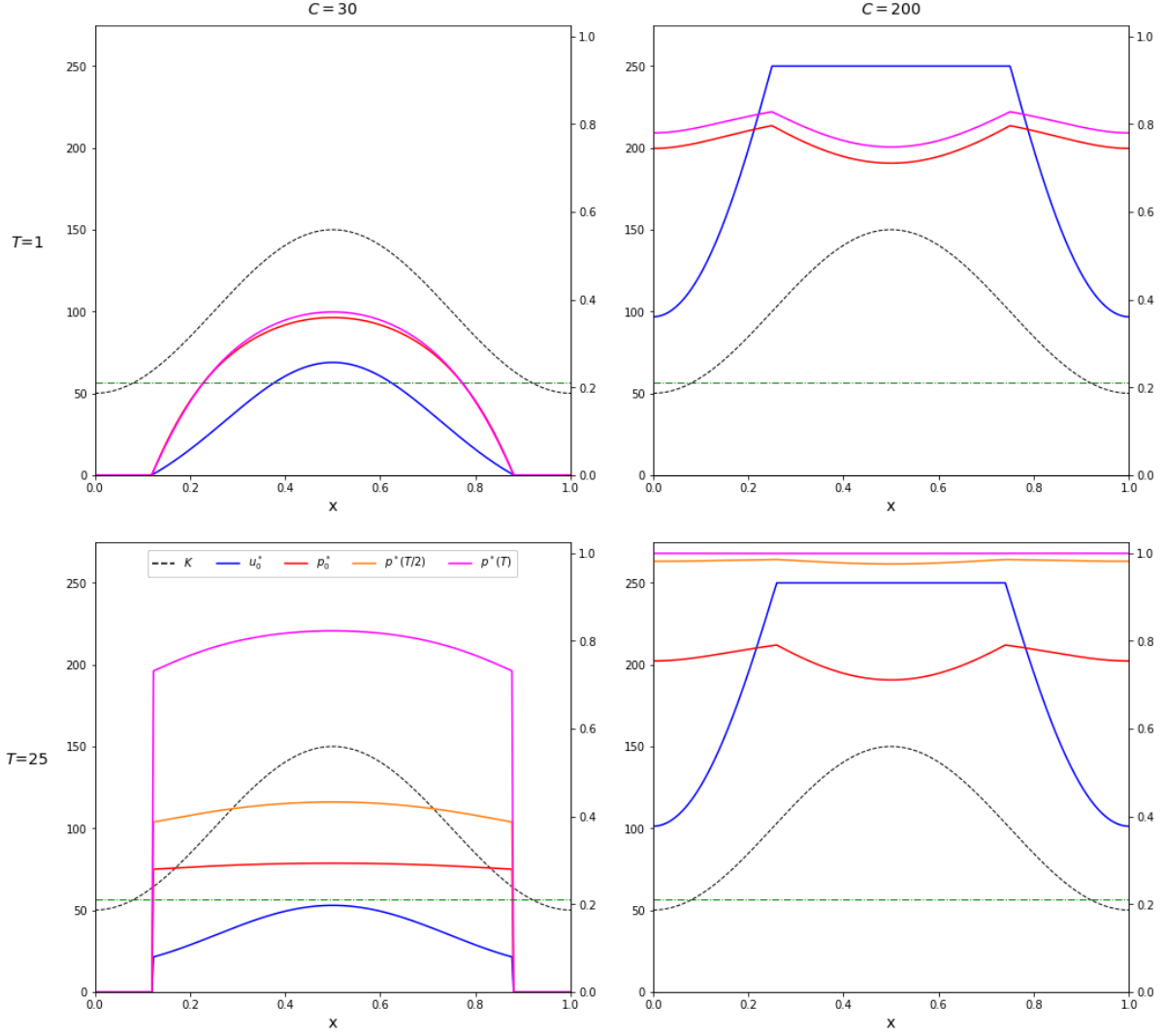


Figure 3: Results for $K(\cdot) = K_S(\cdot)$ for different amount of mosquitoes released, C , and different final times, T . $K(\cdot)$ and $u_0^*(\cdot)$ must be read in the left axis and $p_0^*(\cdot)$, $p^*(T/2)$ and $p^*(T)$, in the right axis. Here p^* stands for the solution of equation (2.13) with initial data $p_0^*(\cdot)$. In green, the line $p = \theta$.

the case, for example, of an urban area close to a wetland. In the case $C = 30$ we can observe again the difference between the short-term and long-term strategies. With $T = 1$ reaching a higher proportion on the left patch is favoured, leaving the second patch untreated. Meanwhile, when the time horizon is increased, it also increases the incentive to release in a wider area above the critical proportion $p_0(x) = \theta$. Therefore the optimal releasing strategy consists in releasing a slightly smaller amount on the left patch in order to release in a certain domain in the right one.

In this case, on the bottom-left graph, we are in the case where $|\tilde{\Omega}_{\lambda_0}| > 0$, and thus the secondary problem must be solved for the values of λ in $[\lambda_0, \lambda_1]$ (see (Q)). Since $K(\cdot)$ is simple, the amount of mosquitoes released and the size of each subdomain of the right patch can be determined almost

explicitely. For a fixed value of $\lambda \in [\lambda_0, \lambda_1]$, the amount of mosquitoes released in the left patch is

$$\int_0^{\frac{|\Omega|}{2}} K(x)G(p_0^*(x)) dx = \int_0^{\frac{|\Omega|}{2}} \frac{3K_0}{2} G\left(\psi_{x,T}^1\left(\frac{2\lambda}{3K_0}\right)\right) dx = \frac{3|\Omega|K_0}{4} G\left(\psi_{x,T}^1\left(\frac{2\lambda}{3K_0}\right)\right),$$

analogously in the right patch $p_0^*(x) = \psi_{x,T}^1\left(\frac{2\lambda}{K_0}\right)$ wherever it is not 0. Therefore the size of the subdomain D , where mosquitoes are released in the right patch is such that

$$C = \frac{3|\Omega|K_0}{4} G\left(\psi_{x,T}^1\left(\frac{2\lambda}{3K_0}\right)\right) + \frac{|D|K_0}{2} G\left(\psi_{x,T}^1\left(\frac{2\lambda}{K_0}\right)\right).$$

This equality can be satisfied for different values of λ and $|D|$. To find the optimal value λ and thus the optimal value of $|D|$, we solve the one-dimensional optimization problem $(\mathcal{Q})$, which in this case reads

$$\begin{aligned} & \min_{\lambda \in [\lambda_0, \lambda_1]} \frac{3|\Omega|K_0^2}{16} ((1 - p^l(T))^2) + \frac{|D|K_0^2}{4} ((1 - p^r(T))^2) \\ &= \min_{\lambda \in [\lambda_0, \lambda_1]} \frac{3|\Omega|}{4} ((1 - p^l(T))^2) + |D|((1 - p^r(T))^2) \\ &= \min_{\lambda \in [\lambda_0, \lambda_1]} \frac{3|\Omega|}{4} ((1 - p^l(T))^2) + \frac{1}{G(p_0^r)} \left(\frac{2C}{K_0} - \frac{3|\Omega|}{2} G(p_0^l) \right) ((1 - p^r(T))^2) \end{aligned}$$

where $p^l(T)$ and $p^r(T)$ solve equation $p'(t) = f(p(t))$ with initial condition $p_0^l = \psi_{x,T}^1\left(\frac{2\lambda}{3K_0}\right)$ and $p_0^r = \psi_{x,T}^1\left(\frac{2\lambda}{K_0}\right)$ respectively. This solution, nonetheless, is what we have been calling unique ‘up to a rearrangement’. As long as the size of the domain where mosquitoes are released is preserved, the solution can be moved on the right half of the domain and still be optimal. In Figure 5 we show another choice for the solution.

Once again, in the case $C = 200$ (right column) the solution does not change significantly when T is increased. We can see how the monotonicity of $u_0^*(\cdot)$ with respect to $K(\cdot)$ is respected, but not for $p_0^*(\cdot)$. In this case, releasing a smaller amount of mosquitoes in the right patch induces a higher initial proportion due to the smaller carrying capacity there.

4.2 The case with diffusion

So far, we have not considered diffusion in the system. Considering that mosquitoes do not disperse simplifies the analysis of the problem, but this comes at the expense of losing realism in the modeling. In this section we present some simulations in which diffusion is taken into account to see how the solutions presented above are modified. The simulations have been carried out using GEKKO (see [8]).

When diffusion is considered, as explained in the beginning of this paper, the equation governing the dynamics of the proportion p is given by (2.7). Once again, we consider a single initial instantaneous release, which modifies equation (2.7) in an analogous way as the one explained for the non-diffusive case.

We are interested in observing how diffusion modifies the results presented so far. We place ourselves in the case with a bigger time horizon, $T = 25$, corresponding with the lower rows of Figures 3 and 4. We show the results for two different diffusion rate values: a smaller one, $D = 0.001$, and a bigger one, $D = 0.02$. In Figure 6, we set $K(\cdot) = K_S(\cdot)$. In the left column, diffusion seems to concentrate the initial distribution of mosquitoes, making it slightly narrower and taller, although

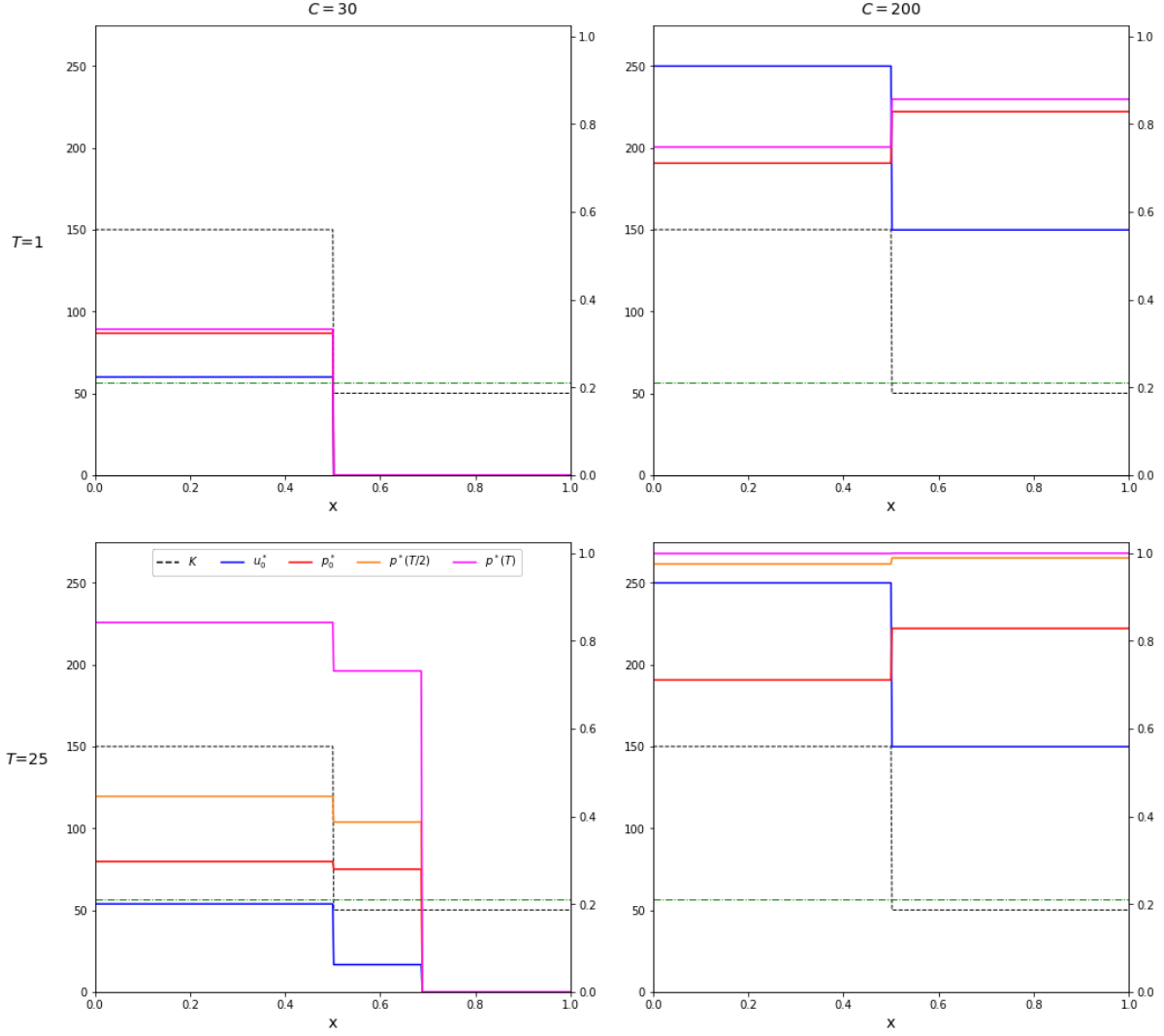


Figure 4: Results for the piece-wise constant carrying capacity, $K(\cdot) = K_P(\cdot)$, for different values of C and T . They are presented in an identical way to Figure 3. The green dash-dotted line represents $p = \theta$.

with a little decrease happening in the center of the release for a small diffusion value (upper left graph). Solutions in this case, nonetheless, seem quite robust to the addition of diffusion for the parameters considered. In the case of a piece-wise constant carrying capacity¹ $K(\cdot) = K_P(\cdot)$ (Figure 7) the addition of diffusion immediately breaks the monotonicity of $u_0^*(\cdot)$ with respect to $K(\cdot)$. This means that, contrary to the case without diffusion, releases do not need to be stronger where there are more wild mosquitoes, and that optimal releasing policies may be more complex. We also observe how mosquitoes are released in a way such that a big density difference is created in the area close to the boundary between patches. As we can see, specially in the bottom left graph, by setting an initial density difference in this boundary, an invasive wave propagates from the patch with a higher

¹Note that K_P as defined is not differentiable and thus ∇K_P and ΔK_P are not defined. Nevertheless, we can always consider a differentiable function $\tilde{K}_P$ such that $\tilde{K}_P(x) = 3K_0/2$ for $x < |\Omega|/2 - \epsilon$, $\tilde{K}_P(x) = K_0/2$ for $x > |\Omega|/2 + \epsilon$ and such that ϵ is smaller than the space step considered in the discretization done by the numerical algorithm implemented to compute the solutions. The carrying capacity in these simulations should be interpreted as the latter case.

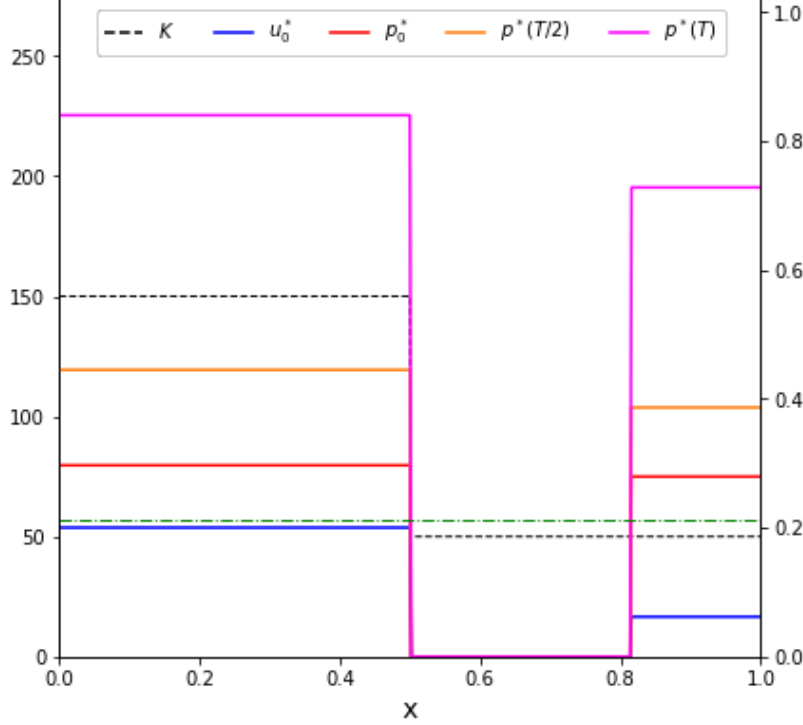


Figure 5: Alternative arrangement of the solution for the case $K(x) = K_P(x)$ with $C = 30$ and $T = 25$ (bottom left graph of Figure 4).

carrying capacity to the other one. It is also noteworthy that in this case a new stationary state appears strictly between $p = 0 \cdot \mathbb{1}_{[0,|\Omega|]}$ and $p = \mathbb{1}_{[0,|\Omega|]}$.

4.2.1 The limit case

We explore in this section the convergence when D tends to zero of solutions for the problem where diffusion is considered, to the solutions of the non-diffusive case. This is a perspective section in the sense that no formal exploration of this convergence is done, but rather the exploration is carried numerically, leaving the formal analysis of the problem for future works.

More rigorously, let us consider $J^\varepsilon(\cdot) := \int_\Omega K(x)^2 (1 - p^\varepsilon(T, x))^2 dx$, where p^ε solves

$$\begin{cases} \partial_t p^\varepsilon - \varepsilon D \Delta p^\varepsilon - 2\varepsilon D \frac{\nabla p^\varepsilon \cdot \nabla K}{K} = f(p^\varepsilon) - \varepsilon D \frac{\Delta K}{K} \psi(p^\varepsilon), & t > 0, \\ \partial_\nu p^\varepsilon = 0, & t > 0, x \in \partial\Omega, \quad p^\varepsilon(0^+, x) = p_0^\varepsilon, \quad x \in \Omega, \end{cases} \quad (4.37)$$

and where p_0^ε is a solution to the problem $\min_{p_0 \in \mathcal{P}_{0,C,M}} J^\varepsilon(p_0)$. Note that system (4.37) is the result of considering an impulsive release at initial time, $u(t, x) = u_0(x)\delta_0(t)$, in equation (2.7), in case $D > 0$. Let us also consider, p_0^* a solution to problem $(\mathcal{P}_{p_0})$, with p_0^* being the initial condition of equation (2.13). The question we want to answer is: do we have convergence of the minimizers for the problem with diffusion to the minimizers of the problem without diffusion? Or more precisely, can we say that $\lim_{\varepsilon \rightarrow 0} p_0^\varepsilon = p_0^*$ in some sense?

To explore this convergence we compare the results of the simulations obtained with our ad-hoc algorithm, developed exploiting theorems 3.6 and 3.8 (case $D = 0$), with solutions obtained using GEKKO for the diffusive case with an ever decreasing D . Let us consider the following carrying

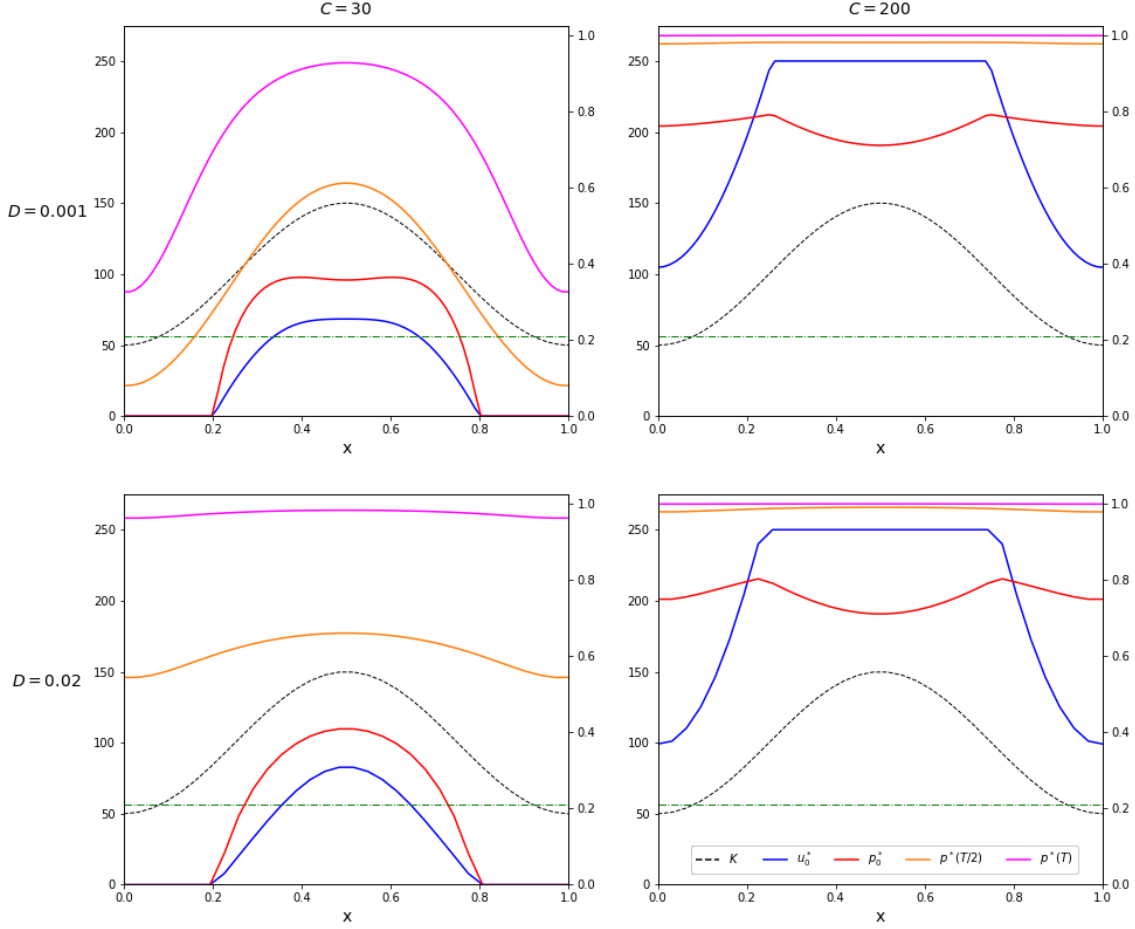


Figure 6: Results for $K(\cdot) = K_S(\cdot)$ for the diffusive system. For different diffusion rates, $D \in \{0.001, 0.02\}$, total amount of mosquitoes $C \in \{30, 200\}$ and final time $T = 25$.

capacity: $K_A(x) = K_0 \left[1 + \frac{1}{\pi} \arctan \left(-10 \left(x - \frac{|\Omega|}{2} \right) \right) \right]$. This carrying capacity models an uneven distribution of mosquitoes, similar to the two-patch situation, but with a smooth transition between them. Again, $\int_{\Omega} K_A(x) dx = K_0 |\Omega|$.

We show the results for both cases, $T \leq T_0$ and $T > T_0$. $T_0 \approx 3.51$ for the parameters of Table 1. In Figure 8 we show the results for $T = 1$. Both in case $D = 5 \times 10^{-2}$ and $D = 0$, releases are stronger in the part of the domain where K is higher, transitioning to zero in the part of the domain where K is lower. Nonetheless, in case $D = 5 \times 10^{-2}$ the tail of this transition is noticeably shorter than in the $D = 0$ case. We observe how, as D decreases, solutions transition from one configuration to the other, with a slight perturbation for $D = 5 \times 10^{-3}$. In case $T > T_0$ (Figure 9) the difference between $D = 5 \times 10^{-2}$ and $D = 0$ is much more notable. Indeed, the solution for $D = 5 \times 10^{-2}$ is almost indistinguishable from the corresponding solution for $T = 1$, while in case $D = 0$ the solution is flatter and wider, with a sharper edge (the reasons for this phenomenon have

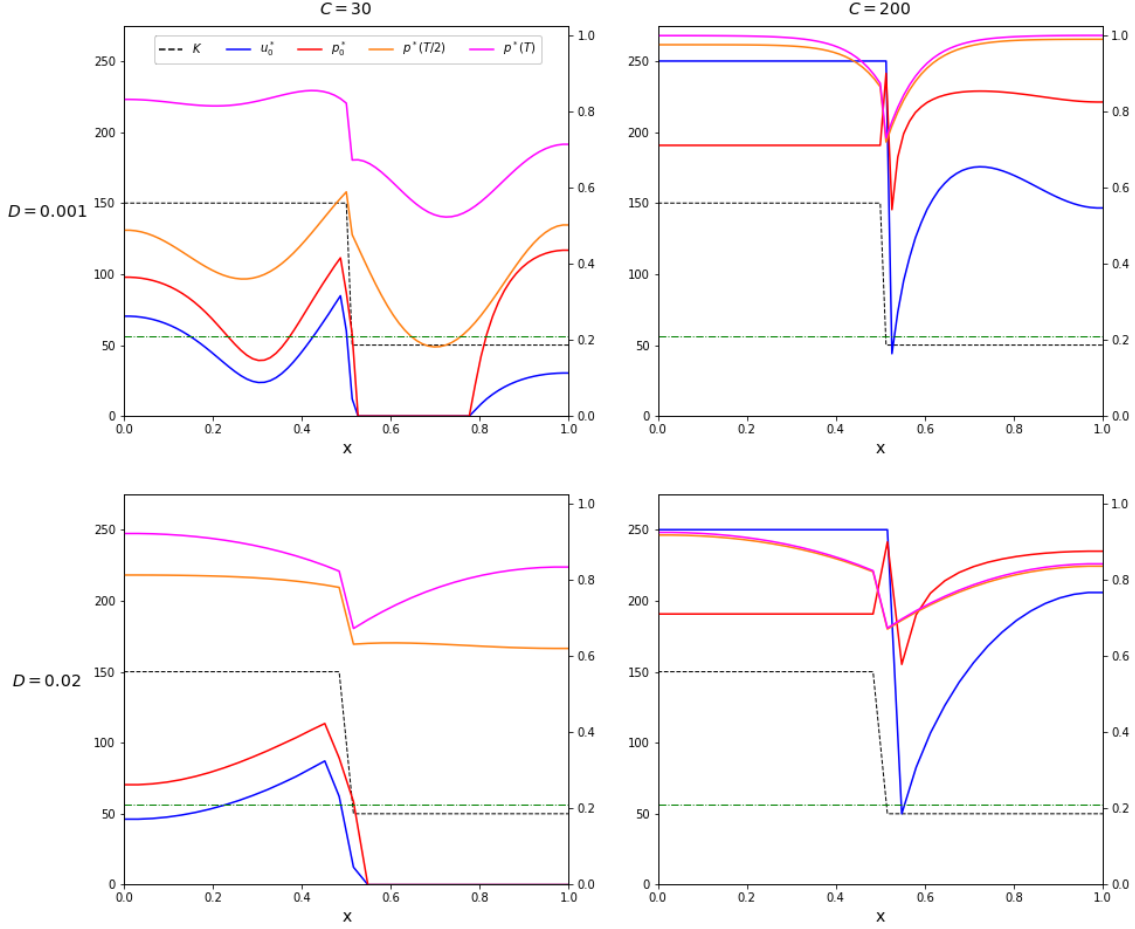


Figure 7: Results for a piece-wise carrying capacity $K(\cdot) = K_P(\cdot)$ for the system with diffusion. Diffusion rates $D \in \{0.001, 0.02\}$ and total amount of mosquitoes $C \in \{30, 200\}$. In green, the line $p = \theta$.

already been discussed in Section 4.1). Nevertheless, as D decreases, we also observe how solutions are modified. At $D = 5 \times 10^{-5}$, the solution is remarkably similar to the case $D = 0$, perturbed only at the edge of the domain and in the transition to zero.

In conclusion, simulations suggest that, indeed, solutions converge in some sense when the diffusion rate tends to zero. That means that, despite the full problem being very hard to study, if the active motion of mosquitoes is sufficiently small which has been described to be the case in certain field releases (see, for instance, [16, 25]), the optimal distribution of a single release at initial time of *Wolbachia*-infected mosquitoes can be approximated by neglecting this motion completely without a big loss of precision. Therefore, beyond being a first step towards more sophisticated studies, our results may be meaningful under certain conditions despite the simplifications performed.

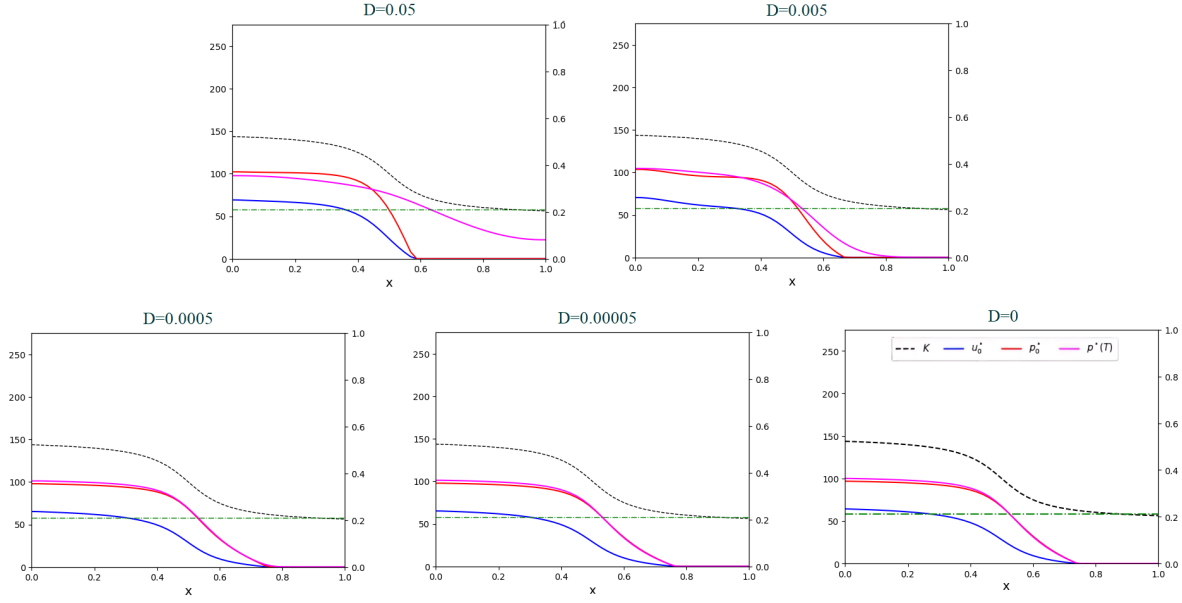


Figure 8: Results for $K(\cdot) = K_A(\cdot)$ in the case $T \leq T_0$. $C = 30$ and $T = 1$ in all the simulations and $D \in \{0, 5 \times 10^{-5}, 5 \times 10^{-4}, 5 \times 10^{-3}, 5 \times 10^{-2}\}$ decreases from left to right, and top to bottom.

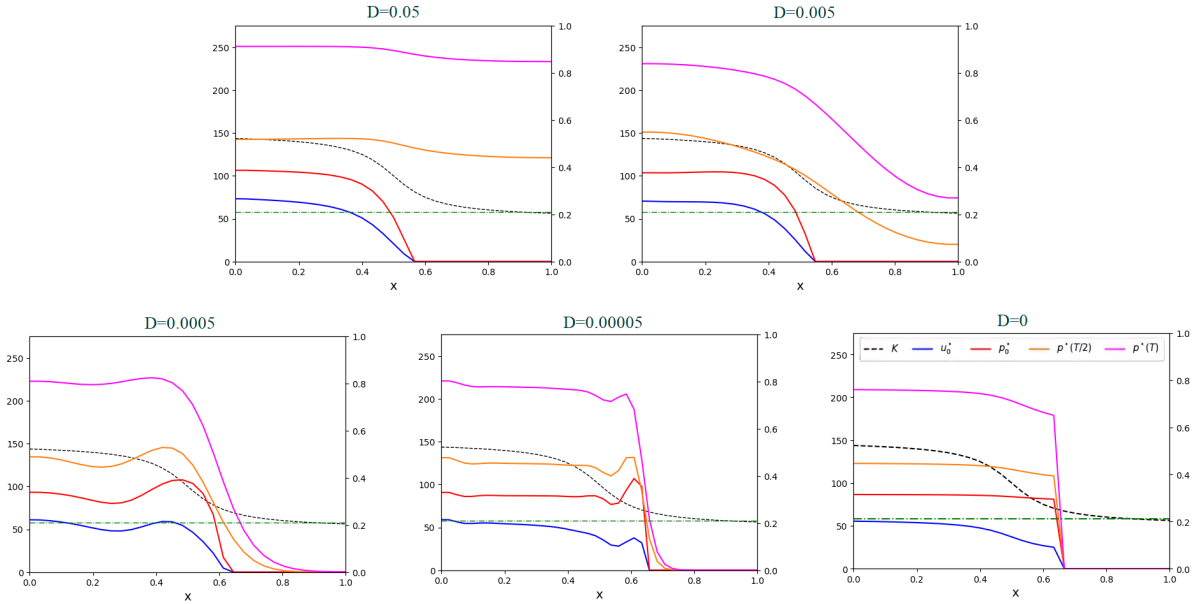


Figure 9: Results for $K(\cdot) = K_A(\cdot)$ in the case $T > T_0$. $C = 30$ and $T = 20$ in all the simulations and $D \in \{0, 5 \times 10^{-5}, 5 \times 10^{-4}, 5 \times 10^{-3}, 5 \times 10^{-2}\}$ decreases from left to right, and top to bottom.

4.3 2D simulations

The method developed in this work can be applied in any dimension. Problems $(\mathcal{P}_{u_0})$ and $(\mathcal{P}_{p_0})$ can be solved using Theorems 3.6 and 3.8 or, at least, reduced to the one-dimensional optimization problem $(\mathcal{Q})$, independently of the number of spatial dimensions considered. For a real application,

nonetheless, the most interesting case is 2D.

Despite not presenting any novelties conceptually speaking, to illustrate the potential of the results, we show also two more simulations done in a 2D setting. We considered the following carrying capacity

$$K_{2D}(x, y) = K_0 \left(1 - \frac{1}{6} \cos \left(\frac{2\pi x}{L_x} \right) - \frac{1}{3} \cos \left(\frac{2\pi y}{L_y} \right) \right).$$

As in the case with K_S , K_{2D} models a scenario with a higher concentration of mosquitoes towards the center of the domain and a smaller one towards the boundaries. Nevertheless, note that K_{2D} is not radially symmetric.

For the simulations we took $\Omega = [0, L_x] \times [0, L_y]$, with $L_x = L_y = 1$. Once again, $\int_{\Omega} K_{2D}(x, y) dx dy = K_0 |\Omega|$ for the chosen parameters. The results of the simulations can be seen in Figure 10. We portray only the case $T = 25$ for two values of C . Results match the intuition one can have from the related 1D case $K(\cdot) = K_S(\cdot)$. When a small amount of mosquitoes is considered, $C = 30$ the solution is flat and wide to surpass the critical proportion $p_0 = \theta$ in a bigger area, since the proportion of *Wolbachia*-infected mosquitoes will naturally increase in those places. Also, $u_0^*(x) = 0$ outside of this area for the reasons already exposed. On the other hand when a bigger amount of mosquitoes is considered, $C = 200$, the solution is bigger than $p_0 = \theta$ everywhere and varies more rapidly, being higher where the carrying capacity is higher, but flattening out when $u_0^* = M$ is reached.

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Appendix

A Numerical exploration of the parameter space for Hypothesis $\mathcal{H}$

Proposition A.1 *Assuming $d_1 \leq d_2 \leq b_2^0 \leq b_1^0$ and $0 < \theta < 1$, then f'' admits a single zero in $(0, 1)$.*

Proof. The existence of a zero of f'' is straightforward to prove: we have $f(0) = f(\theta) = f(1) = 0$, thus thanks to Rolle's theorem, there exist two zeros of f' in $(0, 1)$. Then, applying again Rolle's theorem, we prove the existence of some zero θ_2 of f'' , lying in between the two zeros of f' . Thus, $\theta_2 \in (0, 1)$ in all generality.

Let us now prove the uniqueness of the zero of f'' in $[0, 1]$. By computing the rational function f'' , we see that $\{f'' = 0\} = \{R = 0\}$ where, denoting $\kappa := 1 + s_h - \frac{b_2^0}{b_1^0} > 0$,

$$\begin{aligned} R(p) &:= (s_h - s_h^2 \theta - \kappa^2 + \kappa s_h + \kappa s_h \theta) p^3 + 3(\kappa - s_h - s_h \theta) p^2 - 3(1 - s_h \theta) p + \theta - \kappa \theta + 1 \\ &= \overline{A} p^3 + \overline{B} p^2 + \overline{C} p + \overline{D} \end{aligned}$$

Thus to conclude, it is enough to prove that R has a unique zero in $[0, 1]$. Because $d_1 \leq d_2 \leq b_2^0 \leq b_1^0$ holds, we find that

$$\overline{A} > 0, \quad \overline{B} \leq 0, \quad \overline{C} < 0, \quad \overline{D} > 0.$$

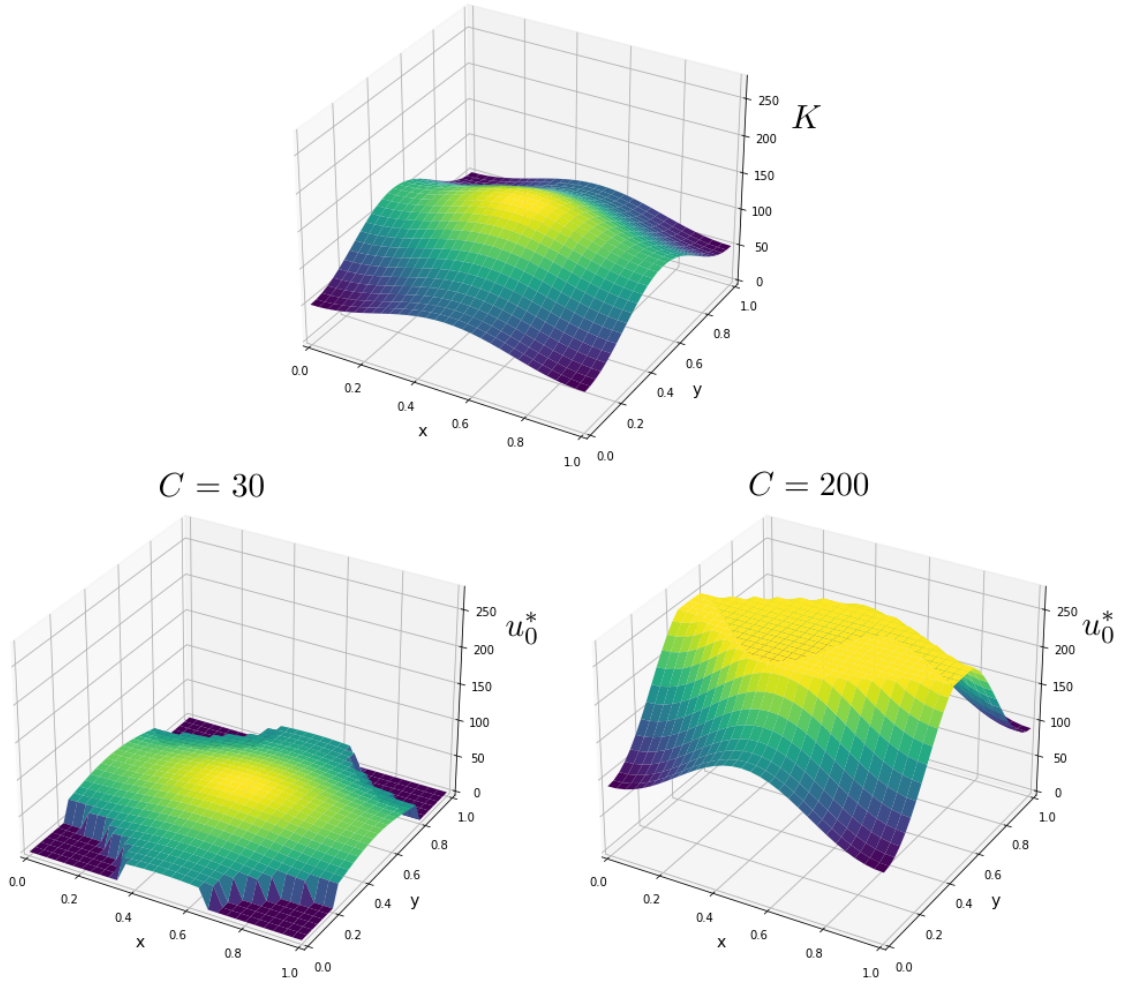


Figure 10: Results in a 2D setting, $K = K_{2D}$, $T = 25$ and $C \in \{30, 200\}$. In green, the level corresponding to $p = \theta$.

Then, using Descarte's rule of signs, we find that R has zero or two positive roots. However, since f'' has at least one zero in $(0, 1)$, so does R , so that R admits exactly two positive roots. Meanwhile, applying the rule to $p \mapsto R(-p)$ implies that R has one negative root.

Now, we set $S(p) = R(p + 1)$. In particular, the leading coefficient of S is $\bar{A} > 0$ while one can prove that

$$S(0) = \bar{A} + \bar{B} + \bar{C} + \bar{D} < 0.$$

Clearly S has three real roots, and their product is given by $-\frac{S(0)}{\bar{A}} > 0$. However, S has at least one negative root since R also does. Since the product of all three roots of S is positive, S has exactly two negative roots and one positive root. As a result, R has exactly one root in $[0, 1]$, and the conclusion follows. $\square$

This section is devoted to a numerical exploration of the space of parameters, in order to establish the validity of Hypothesis $(\mathcal{H})$. For the exploration, we normalize the parameters assuming $b_1^0 = 1$. The results are presented in Figures 11 and 12. To produce these images for a given pair of values

(s_h, b_2) , two random values for d_2 and d_1 are chosen, such that $d_1 \leq d_2 \leq b_2^0 \leq b_1^0 = 1$. If the randomly generated set of parameters are such that $\theta \notin (0,1)$, the set is discarded. If indeed $0 < \theta < 1$, then the Hypothesis ($\mathcal{H}$) is tested. In blue, are the values of the parameters for which Hypothesis ($\mathcal{H}$) is satisfied. In red, the values for which is not.

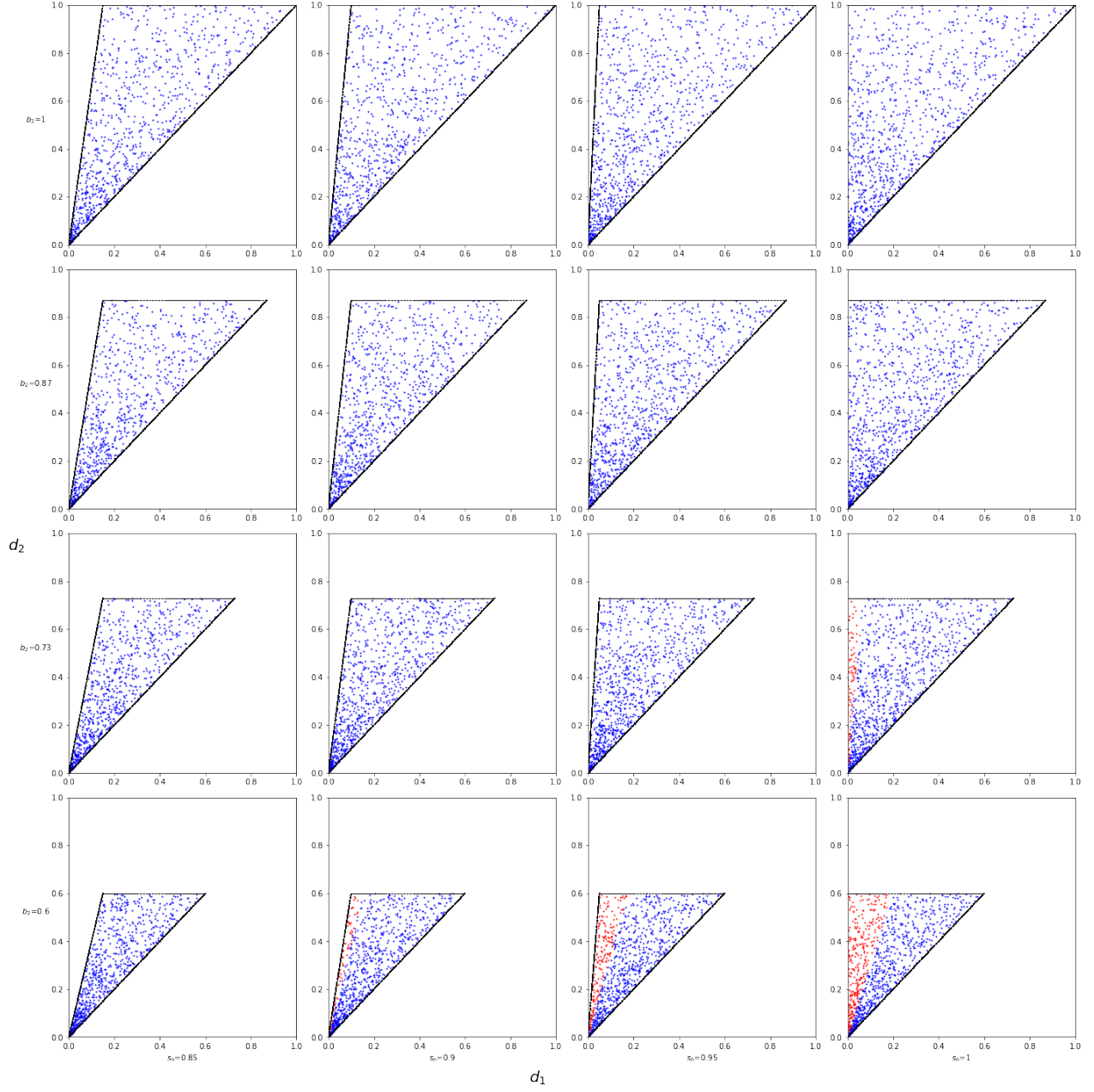


Figure 11: Hypothesis ($\mathcal{H}$) tested for different parameter values. All pictures have d_1 in the x -axis and d_2 in the y -axis. In this image, $s_h \in \{0.5, 0.67, 0.83, 1\}$ increases from left to right and $b_2 \in \{0.6, 0.73, 0.87, 1\}$ decreases from top to bottom. Blue dots mean Hypothesis ($\mathcal{H}$) is satisfied for those parameters, while red dots mean it is not.

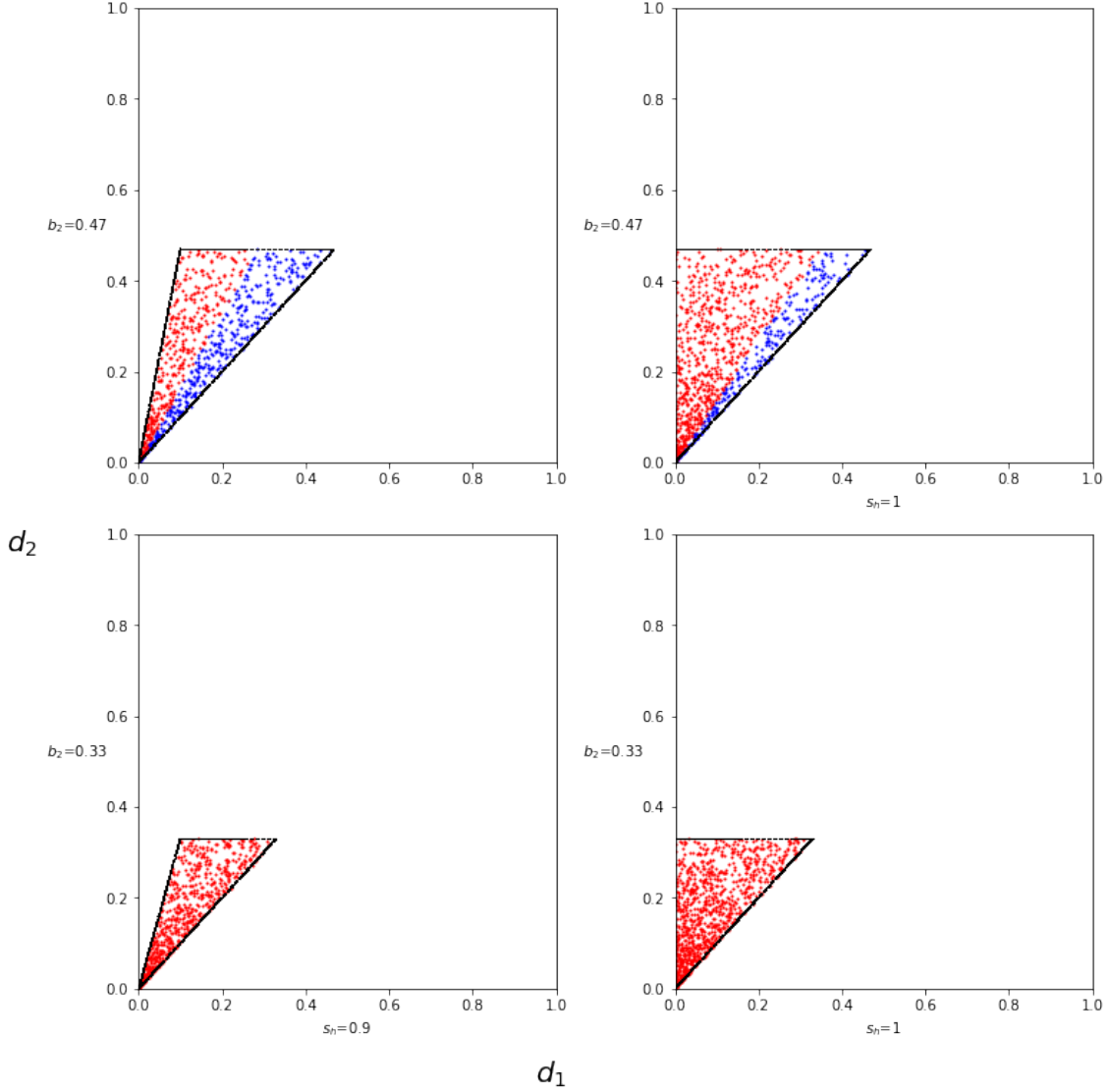


Figure 12: Hypothesis ($\mathcal{H}$) tested for small values of b_2 and high values of s_h . All pictures have d_1 in the x -axis and d_2 in the y -axis. In this image, $s_h \in \{0.9, 1\}$ increases from left to right and $b_2^0 \in \{0.33, 0.47\}$ decreases from top to bottom.

The condition $\theta \in (0, 1)$ can be written in terms of the other parameters,

$$0 < \theta < 1 \Leftrightarrow 0 < 1 - \frac{d_1 b_2^0}{d_2 b_1^0} < s_h \Leftrightarrow 1 > \frac{d_1 b_2^0}{d_2 b_1^0} > 1 - s_h \Leftrightarrow \frac{d_2}{d_1} > \frac{b_2^0}{b_1^0} > \frac{d_2}{d_1} (1 - s_h).$$

Since $b_2^0 \leq b_1^0$ and $d_1 \leq d_2$, this means that if $b_2^0 \leq 1 - s_h$, necessarily $\theta > 1$, thus these values can be excluded from the exploration. Indeed, the black lines wrapping the dots in Figures 11 and 12 are $d_2 = d_1$, $b_2^0 = 1$ and $b_2^0 = 1 - s_h$.

As we can see in Figure 11 and 12, Hypothesis $\mathcal{H}$ is satisfied by most of the parameters of the parameter space. For high values of b_2^0 it is always satisfied (also for those values not shown in Figure 11). As b_2^0 decreases, red dots appear only when we have high values of s_h and $d_2 \gg d_1$. Only for high values of s_h and small values of b_2^0 the red dots dominate the picture. In a realistic

scenario, based on the values for the parameters found in the literature [6, 13] we expect a high value of s_h and b_2^0 and smaller values of d_2 and d_1 , for which hypothesis $(\mathcal{H})$ is satisfied. The values of the parameters not satisfying Hypothesis $(\mathcal{H})$ would represent a particular strain of *Wolbachia* that, in a certain variety of mosquito, would produce a high CI rate and a big penalty on their fertility, which, a priori, is not impossible.

B Comments on the existence of solutions for problem $(\mathcal{P}_{p_0})$

In this work, we do not establish the existence of minimizers for problem $(\mathcal{P}_{p_0})$ in all generality. Despite not exploring the matter separately, we do prove the existence of solutions in the case $T \leq T_0$ in Theorem 3.6. Indeed, in this theorem, we provide necessary conditions that solutions must satisfy. Then, in its proof, we exploit these conditions to narrow down the solution space, obtaining a single candidate to solution, p_0^* . Since the constructed p_0^* is the only candidate solution and we prove that $p_0^* \in \mathcal{P}_{0,C,M}$, we conclude not only the existence but also the uniqueness of the solution (as already mentioned, up to a rearrangement in case $K(\cdot)$ is piecewise constant). Conversely, for the case $T > T_0$, the candidate solution we construct in the proof of Theorem 3.8 by exploiting the necessary optimality conditions is not unique, but rather a whole family of candidates depending on a parameter $\lambda^* \in [\lambda_0, \lambda_1]$. Hence we cannot conclude the existence of solutions from this result. On top, both theorems only hold under hypothesis $(\mathcal{H})$.

Although tackling comprehensively the existence of minimizers is beyond the scope of this work, we present here a partial result settling the existence in 1D, for all $T > 0$, for the case where $K(\cdot)$ is piecewise constant. This proof, nevertheless, cannot be easily extended to the general case.

Proposition B.1 *Let K be a piece-wise constant function, with $x \in \Omega \subset \mathbb{R}$. Then, there exists $p_0^* \in \mathcal{P}_{0,C,M}$ solving problem $(\mathcal{P}_{p_0})$.*

Proof. Let us write $K(x) = \sum_{i=1}^n K_i \mathbb{1}_{[x_{i-1}, x_i]}$. We place ourselves in one of the intervals $[x_{i-1}, x_i]$, where $K(\cdot)$ is constant.

Let us consider any function $p_0 \in \mathcal{P}_{0,C,M}$ in this interval. We claim there exists a monotonic (decreasing or increasing) rearrangement of p_0 , that we will denote $\hat{p}_0$ such that $\hat{p}_0 \in \mathcal{P}_{0,C,U}$. To define this rearrangement, let us introduce, in a given interval $[x_{i-1}, x_i]$,

$$\mu(s) := |\{x \in [x_{i-1}, x_i] : p_0(x) > s\}|.$$

Then, we define $\hat{p}_0$ in that interval as

$$\hat{p}_0(x) := \inf\{s \in [0, 1] : \mu(s) \leq x\}.$$

The fact that such a rearrangement will respect $0 \leq \hat{p}_0 \leq G^{-1}(M/K(x))$ is trivial since rearranging a function does not change its maximums or minimums (see [27]). Suppose $\int_{\Omega} K(x)G(p_0(x)) dx \leq C$, then

$$\int_{\Omega} K(x)G(\hat{p}_0(x)) dx = \int_{\Omega \setminus [x_{i-1}, x_i]} K(x)G(p_0(x)) dx + K_i \int_{x_{i-1}}^{x_i} G(\hat{p}_0(x)) dx$$

for every $i \in \{1, \dots, n\}$. G is a continuous function, hence it is measurable. Therefore, since G is non-negative and measurable we have

$$\int_{x_{i-1}}^{x_i} G(p_0(x)) dx = \int_{x_{i-1}}^{x_i} G(\hat{p}_0(x)) dx$$

by equimeasurability of the rearrangement.

This implies that if $\int_{\Omega} K(x)G(p_0(x)) dx \leq C$, then $\int_{\Omega} K(x)G(\hat{p}_0(x)) dx \leq C$. This reasoning can be easily extended to all of the subintervals. Therefore we have proved that $\mathcal{P}_{0,C,M}$ is stable under rearrangements.

Let us define now

$$\hat{\mathcal{P}}_{0,C,M} := \{\hat{p}_0 \in \mathcal{P}_{0,C,M} \mid \hat{p}_0 \text{ is monotonic in } [x_{i-1}, x_i], i = 1, \dots, n\}.$$

Observe that

$$\min_{p_0 \in \mathcal{P}_{0,C,M}} J^0(p_0) = \min_{\hat{p}_0 \in \hat{\mathcal{P}}_{0,C,M}} J^0(\hat{p}_0),$$

with J^0 defined by (2.11). Indeed, if p_0^* is minimizer of J^0 in $\mathcal{P}_{0,C,M}$, then

$$\int_{\Omega} K(x)^2(1 - \hat{p}(T, x))^2 dx = \int_{\Omega \setminus [x_{i-1}, x_i]} K(x)^2(1 - p(T, x))^2 dx + K_i^2 \int_{x_{i-1}}^{x_i} (1 - \hat{p}(T, x))^2 dx,$$

for every $i \in \{1, \dots, n\}$, where we are denoting by $\hat{p}(t, x)$ the solution to equation (2.13) with initial condition $\hat{p}_0(x)$.

We can follow the same reasoning as before to prove that

$$\int_{x_{i-1}}^{x_i} (1 - p(T, x))^2 dx = \int_{x_{i-1}}^{x_i} (1 - \hat{p}(T, x))^2 dx$$

for every $i \in \{1, \dots, n\}$. To see this clearly, we can write $p(t, x)$ as a function of its initial condition by realising that $p(t, x)$ can be written as

$$\frac{\partial}{\partial t} p(t, x) = f(p(t, x)) \Rightarrow \int_0^p \frac{d\nu}{f(\nu)} = \int_0^t ds = t.$$

Defining $F(p)$ as the antiderivative of $1/f(p)$ vanishing at 0, $F(p) := \int_0^p \frac{d\nu}{f(\nu)}$, we can write

$$F(p(T, x)) = F(p_0(x)) + T \Rightarrow p(T, x) = F^{-1}(F(p_0(x)) + T). \quad (\text{B.38})$$

Both F and its inverse are continuous functions and thus, so it is its composition. Therefore, $p(T, \cdot)$ is also a measurable function of $p_0(x)$ and since it is non-negative both integrals are equal.

This implies that, if there exists a solution monotonic by intervals, $\hat{p}_0^* \in \hat{\mathcal{P}}_{0,C,M}$, there must also exist a solution in $\mathcal{P}_{0,C,M}$. Thus, we restrict our analysis to the first kind of functions.

Let us consider a minimizing sequence $(\hat{p}_0^n)_{n \in \mathbb{N}} \in \hat{\mathcal{P}}_{0,C,M}$ for problem $(\mathcal{P}_{p_0})$. We know it exists since $\hat{\mathcal{P}}_{0,C,M}$ is non-empty. Due to the fact that for all $n \in \mathbb{N}$, $0 \leq \hat{p}_0^n(x) \leq G^{-1}(M/K(x))$ a.e. in Ω and using the monotonicity of $\hat{p}_0^n$ on each interval (x_{i-1}, x_i) , we deduce from Helly's selection theorem (see [28]) that $(\hat{p}_0^n)_{n \in \mathbb{N}}$ converges pointwisely to an element $\hat{p}_0^*$, up to a subsequence.

Basic properties of pointwise convergence lead us to conclude that $0 \leq \hat{p}_0^*(x) \leq G^{-1}(M/K(x))$ a.e. in Ω . Moreover, according to the Lebesgue dominated convergence theorem, one has

$$\int_{\Omega} K(x)G(\hat{p}_0^*(x)) dx = \lim_{n \rightarrow \infty} \int_{\Omega} K(x)G(\hat{p}_0^n(x)) dx = \lim_{n \rightarrow \infty} \langle K(x)G(\hat{p}_0^n(x)), 1 \rangle_{L^\infty, L^1} \leq C.$$

Indeed, we recall that $K(\cdot)$ is piecewise constant and thus it does not affect the convergence properties of the sequence under the integral. Therefore, $\hat{p}_0^* \in \hat{\mathcal{P}}_{0,C,M}$. A similar reasoning shows that the sequence $(F^{-1}(F(\hat{p}_0^n(x)) + T))_{n \in \mathbb{N}}$ converges almost everywhere in Ω and therefore, we have

$$\lim_{n \rightarrow \infty} J^0(\hat{p}_0^n(x)) = J^0(\hat{p}_0^*(x)),$$

according to (B.38). It follows that $\hat{p}_0^*$ is indeed a solution to problem $(\mathcal{P}_{p_0})$.

□

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